Geometry

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Geometry in the Primary Class

Before the child is given geometry at the elementary level, he has had remote preparation for it in the home and the primary class. He has sensorially experienced the shapes around him and may have been given the correct names for these. The sensorial material in the primary class is structured to help the child clarify and classify these impressions.

The geometry cabinet is designed to give the child experience with plane geometry form and the appropriate language for the figures he’s encountered. The child begins with a few contrasting forms and gradually works through the draws to refine his discrimination and become aware of small differences between figures. His work with the card material takes his experience a step further until the child forms an intuitive knowledge of figures. The child is gradually led from recognizing shape with his muscular memory to the point of recognizing them visually. The geometric cabinet also subconsciously prepares the child for the discrimination of form.

The child forms a basic knowledge of solid forms through his work with the geometric solids. By associating them with their bases, the child can begin to make the connection between the solid and it’s two-dimensional shape. Through his use of the constructive triangles, the child is able to discover the constructive aspect of triangles as he puts them together to create new shapes. The superimposed geometric figures are designed to enable the child, through experience, to find out the relationship between certain figures. The child’s work with the metal insets helps him to gain further knowledge of what makes up the basic shapes. By drawing around the figures, the child is able to make the shapes himself and with the later exercises begins to see more detail between shapes and their relation to each other. The small metal insets further this experience and helps the child become aware of different area shapes occupy. With the classified card material the child is given the language for the various parts of geometric shapes.

Some children in the primary class will have had the opportunity to use a ruler to measure. Others may have been shown how to use a compass and may know a little bit about angles and how to measure them using a protractor. Some nomenclature may be known through the children’s work with the small metal insets, the classified nomenclature cards and his hand work with the compass, protractor etc. A few children may have had simplified versions of the equal, similar and equivalent figure lessons.

The child entering the elementary class will at least have had some sensorial based work, will be familiar with the geometric cabinet and cards, will know the basic language for a circle, square, rectangle, triangle and at least some names of regular polygons. He will have had experience with the geometric solids and their bases and should be familiar enough with them to make the connection from the solid to the plane shape. (Language is helpful for this connection.) He will have had constructive triangle work and an opportunity to work with the superimposed geometric figures. He will also have done some designing work with the large metal insets.

Assessing What the Child Knows

It is important to see our work as a continuation of the child’s experience with geometry aiding him in making a connection to the shapes he sees around him. All children have experienced a long period of remote preparation and their knowledge of shapes will vary considerably. It is necessary when the children first come to find out which shapes they know. This should be done informally through games relating the shapes to the environment such as listing the shapes they see, collecting pictures for a collage etc. Informal games can also be used to give the children information they don’t already have and provide a basis for further exploration.
The study of geometry is particularly suited to the child at this plane of development because of his investigatory nature and thrust for knowledge. The geometry materials he is presented are creative in nature and provide ample opportunities for the child to create his own abstractions. The laws that govern geometry will be given to the child in secondary school with thermo but through demonstrations of the principles the child can be familiarized to these with out the language providing a stimulus for intellectual development and experience with logical reasoning, deduction and in forming abstractions. The manual activity the child uses aids his intellectual development establishing a basis for later abstracting.

A general outline for the chapters is as follows:
1. Study of Lines
2. Study of Angles
3. Polygons
4. Equivalence, similarity & congruence
5. Area of Plane Figures
6. Solid Geometry

Presentations to the children are not given in that order but in a manner which creates interest in the children. The sequence will vary with different children and not all areas need to be covered as long as the appropriate public school curriculum is.

Historical Notes

Historical Outline of Geometry

The Egyptians
1. Gave geometry it’s name after the “earth measurements” made every year to mark their fields after the Nile flooded.
2. Had only a practical interest in geometry; used it because it worked but didn’t know why or how.
3. Used an isosceles triangle with a weight hung from it’s vertex as a level.
4. Marked their fields with a right angle using a knotted rope forming a right angled scalene triangle. (knotted in a 3:4:5 ratio)

Ancient Greeks

Thales (600B.C.)
1. Gave geometry it’s present name; interpreted ‘earth measurement’ into Greek, geos for earth, metron for measure.
2. Became interested in Egyptian practical use of geometry; lead him to begin a deductive science of geometry.
3. Used his knowledge of geometry to calculate the height of a pyramid and the distance of a ship from shore -a practice of application not common among mathematicians of his day.
Pythagoras (540 B.C.)

1. Studied under Thales; visited Egypt to study the ‘earth measurements’ of the Egyptian priests.
2. Developed a school of mathematics not interested in the practical use of geometry rather the abstract side. (propositions or thermos)
3. Known for his theorem: the area of a square drawn on the longest side of a right triangle is always equal to the sum” of the areas of squares drawn on the other two sides.
4. Plato (390 B.C.)
5. Believed any man who wished to become a leader should be trained in mathematics; not for practical purposes, but for training the mind.
6. Felt that amusement and pleasure should be combined with instruction to make the subject interesting.

Introduction Story

A long, long time ago, when people were just learning about mathematics and what numbers could do, there lived a group of people and they called themselves Pythagoreans. They belonged to a school of mathematical philosophy which was named after Pythagoras, a man so well known for his ideas that his name can still be found in every book on geometry today.

The Pythagoreans weren’t interested in finding practical uses for geometry as the ancient Egyptians had been by using it to mark their fields and measuring if their floors were even, but wanted to study and prove why geometry works as it does. While the Egyptians knew they could mark a right angle in the field by knotting a rope in a particular fashion and having their slaves hold the robe at the knots to form a triangle, they did not know or even care why this worked every time. The Pythagoreans, on the other hand, didn’t try to think of ways they could apply this idea, but set out to prove why these and other geometry facts happened.

Among the discoveries made by the Pythagoreans was that numbers could be odd or even. As Greek society at the time was male-denominated, the odd numbers, which they felt to be better than the evens, were male numbers and were divine. Even numbers were female and considered to be earthly. So the odd numbers were considered lucky ones while the even numbers had no choice but to be unlucky. This idea lasted for many years and could even be found in a Shakespeare play in a line stating, “This is the third time, I hope good luck lies odd numbers… They say there is divinity in odd numbers, either in nativity, chance, or death.”

Another discovery of the Pythagoreans was that the earth was a sphere. By observing the shadow cast by the earth on the moon, they determined the earth must be round rather than flat as generally thought. Even though Greek mathematicians later accepted this idea, it wasn’t until the 15th and 16th centuries when Columbus and others began to navigate the great oceans that the average person became convinced the earth was spherical.

Either Pythagoras or one of his followers was responsible for discovering the harmonic progression in the musical scale. Other accomplishments include developing a theory of numbers, and searching for an answer for such questions as “Can any two lengths be divided into parts, all which are equal to each other?” or “Can any flat surface be completely filled by repetition of the same figure?”

The followers of Pythagoras continued to function long after his death and were a great influence throughout
the Greek world. At first, they were bound by an oath not to reveal the secrets and teachings of their brotherhood to anyone else, and only passed these on orally to one another. But as time went by, they began to put their work into writing preserving their work to this day.

**Note:** Examples of dodcahedron (“12-faced” solid) and icosahedron (“20-faced” solid) can be brought in and discussed in conjunction with the story as a discovery of the Pythagoreans.
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**Congruent, Similar and Equivalent Figures**

**Materials:**
- Metal square insets or two sets of divided squares as illustrated (note- I have found that circles work well for congruency and similarity.)
- Slips of paper
- Pencil

**Prerequisite:**
- Child must know shapes.
- Child may be familiar with the inset from the primary level.

**Notes:**
- Material can be used to make designs
- Can be shown to a child or group
- **Congruent** means - same shape and same size.
- **Similar** means same shape but different in size.
- **Equivalent** means they are the same size but different shape.
- Equivalency sign - three lines, longer than the equal sign
**Presentation 1: Congruent**

1. Pick up two sections from the divided square and compare them back to back.
2. Ask “Are these the same?”
3. State that when two figures are the same in every way, they are called congruent figures.
4. Ask each child to find two congruent figures, and to state what shape it is.
5. Have the children tell you why they are congruent, and if it is the same in every way.
6. Have children close their eyes; place a figure in front of each, and have them match with a congruent figure.
7. Ask them to find other figures that are the congruent.

**Variations:**

1. Ask a child to trace the figure on a paper.
2. Later, give children the symbol for identical or congruent.

**Presentation 2: Similar**

1. Pick up the whole square and ask what it is called.
2. Pick up a smaller square and ask the same question.
3. State that figures are called similar when they have the same shape and name, but the size is different.
4. Ask the child to find a figure of the same shape but of different size.
5. Ask why is it similar?
6. After they have found a lot of similar shape, give the child the sign for similar

**Variations:**

1. Make charts or booklets of similar and congruent figures.
2. **Give children a printed label stating a number of identical or similar figures and have them find these;** after, ask what each has and how they knew.
3. Do a lot of work on this before moving on.
Presentation 3: Equivalent

1. Review identical and similar figures.
2. Take out the 1 whole square; place the rectangular half over it stating that it is half.
3. Repeat with the triangular half.
4. State that the two halves are equivalent because they have the same size, but a different shape.
5. Give each child a piece and have them find it’s equivalent; ask what they have and how they knew.
6. Repeat procedure with different figures.
7. Give printed slips with all three to locate.
8. Encourage independent work – booklets, charts, etc.
Constructive Triangles

Materials:

- Three Boxes of triangles made of colored wood: A triangular box, a small hexagonal box, and a large hexagonal box
- The material is called “constructive” because the triangle is used to form other rectilinear figures.

Triangular Box

- 1 large gray equilateral triangle without black lines
- 2 equal green scalene right angle triangles with black lines along the longer of the two sides enclosing the right angle.
- 3 equal yellow isosceles obtuse angle triangles with black lines along both sides enclosing the obtuse angle
- 4 equal red equilateral triangles; one with black line on all sides, three with black lines on one side

Small Hexagonal Box

- 6 gray equilateral triangles with black lines on 1 side
- 2 red equilateral triangles with black lines on 1 side
- 3 green equilateral triangles; 2 with black lines on 1 side, 1 with black lines on two sides
- 1 large yellow equilateral triangle with no black lines
- 6 red obtuse angle isosceles triangle with black lines on side opposite the obtuse angle.

Large Hexagonal Box

- 2 red isosceles obtuse angle triangles with black line on side opposite the obtuse angle
- 2 gray isosceles obtuse angle triangles with black line on one of the sides enclosing the obtuse angle
- 6 yellow isosceles obtuse angle triangles; 3 with black lines on side opposite obtuse angle, 3 with black lines on both equal sides.
- 1 large yellow equilateral triangle whose sides equal longest side of other triangle, with black lines on all 3 sides
Example Equivalencies

- a parallelogram ≡ a rectangle ≡ a triangle
- a square ≡ a rhombus
- a right-angled scalene triangle ≡ a hexagon

Triangular Box

Triangular Box Examples

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Small Hexagonal Box

Contents of box: sizes are NOT relative to large hexagonal box illustration

Example Equivalencies:
Large Hexagonal Box

Example Equivalencies:
Triangular Box

Material:
Triangular box

Notes:
• Review – remember how to construct the four triangles
• Triangular box- only one pair of similar figures

Presentation:
1. Gather a small group of children.
2. Lay out the pieces randomly; Have each child find 2 congruent pieces and match to make sure its equal.
3. Ask if they remember how to put them together; have them construct the four triangles.
4. Work with the children in making congruent, similar and equivalent figures.
5. Point out important equivalencies that the children didn’t find on their own.

Small Hexagonal Box

Presentation:
1. Lay out all the pieces; introduce shapes to children who don’t know.
2. Have the children find congruent shapes; ask how they know they are identical. (can superimpose)
3. Ask one child to find a pair of similar figures; ask if there are anymore.
4. Have children find equivalent figures; superimpose to illustrate where appropriate.
5. Demonstrate important ones not found.

Large Hexagonal Box

Presentation:
1. Lay out all the pieces; have the children find congruent shapes; ask how they know they are congruent. (can superimpose)
2. Ask them if there are any similar figures, (there aren’t any.)
3. Have children find equivalent figures; superimpose to illustrate where appropriate.
4. Demonstrate important ones not found.
Combining Boxes

**Prerequisite:**
- Identical, similar and equivalent work
- Triangular box
- Small and large hexagonal box

**Materials:**
- Triangular box
- Small hexagonal box
- Large hexagonal box

**Presentation:**
1. Gather a small group of children.
2. Combine the triangular and small boxes.
3. Lay out the pieces of the boxes randomly
4. Have the children find congruent, similar and equivalent figures.
5. Continue with other combinations of boxes, eventually working with all three to find equivalences.

**Blue Right Scalene Triangles**

**Materials:**
- Small box of 12 right angle scalene triangles

**Presentation:**
1. Present box to children.
2. Have the children make standard shapes using 4, 6 and 12 triangles through experimentation and demonstration as needed.
III  Geometry Sticks

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Presentation:

Exercise Four: Intersecting Lines  p.37

Presentation:
Observations

1. An equilateral triangle will always be an acute triangle.
2. A right triangle is never an obtuse or an acute triangle.
Polygons

Materials:
- Geometry Stick Box
  This box is divided into twelve sections. Ten sections have colored sticks; each set has a different length and color, most sticks have small holes at each end for fastening.
  - 2 cm. – brown
  - 4 cm. – purple
  - 6 cm. – orange
  - 8 cm. – red
  - 10 cm. – black
  - 12 cm. – light brown
  - 14 cm. - pea green
  - 16 cm. – pink
  - 18 cm. – blue
  - 20 cm. – yellow
- Brads
- Tacks
- Right angle-measuring angle
- Weighted string

Presentation: Part One
1. Fasten two sticks together at their ends. Ask the children to make a shape out of it.
2. Ask, “Is it possible to make a shape out of the two sticks?”
3. Have each child take a stick and a brad. Have each connect his/her stick to the original two.
4. Close the shape when all have been attached.
5. State: “With 2 we couldn’t make a shape, but now we have one.”
6. Place the shape on the table; label the inside part the “interior” and the outside the “exterior”.
7. Introduce the shape as a “polygon” meaning ‘many sides’.
8. Take one stick out and don’t reattach. Ask if it’ a polygon now (no, because it’s not closed).
9. State that a polygon is a closed figure made up of many sides that are all straight lines.
Presentation: Part Two

1. Have each child take two sticks and a brad. Have them try again to make a shape stating that maybe they didn’t try hard enough the first time.

2. Ask how few sticks are needed to make a shape. Have them take another stick and try.

3. State that with two sticks we couldn’t make a shape, now with three sticks we can make a shape.

4. Ask what shape it is. If needed, state that it’s a polygon with three sides, which is called triangle.

5. Make another polygon with four sticks. State that a polygon with four sides is called a quadrilateral (Quad means 4 and lateral means side).

6. Give special names for quadrilateral; square, parallelogram, rhombus, rectangle.

7. Do a lot of experimenting, have the children make polygons with more than 4 sides and review what each has made.

8. Have the children group all the polygons together and write a label for each.

9. Mix up shapes, hand each child a label and ask them to find the matching shape. Ask what each has, how many sides and if it is still a polygon.

Additional Activities:

1. Have each child hold up a triangle in one hand, and another shape in other; see what happens. Try a different shape with the triangle; discuss that the triangle is the only one that keeps its shape.

2. In a group, give one child a triangle, one a quadrilateral, one a polygon. Ask what each has and how they know it was a polygon. Replace one side of the polygon with a curved piece. Ask if it’s still a polygon and why/why not. (only a polygon when enclosed with straight lines)

3. Follow-Up: Have children search around the room to find polygons; draw shapes; older children can construct using a ruler.

Notes:

- 3 sides – Triangle
- 4 sides – Quadrilateral
- 5 sides – Pentagon angle 108 degrees
- 6 sides – Hexagon angle 120 degrees
- 7 sides – Heptagon angle 128.57142 degrees
- 8 sides – Octagon angle 135 degrees
- 9 sides – Nonagon angle 140 degrees
- 10 sides – Decagon
- 12 phase solid – Dodecahedron
- 20 phase solid – Icosahedron
Angles

Exercise One: Complete and Straight Angles

Materials:
- Geometry sticks box
- Cork board
- Paper (newsprint)
- Pencil

Presentation:
1. Set a piece of paper on the corkboard and secure it with tacks.
2. Pick a long stick with a hole at each end and a shorter stick with several holes.
3. Place the shorter stick on top of the longer stick and secure both ends with a tack in the middle of the corkboard.
4. Place the tip of the pencil at the hole at end of the top shorter stick.
5. Draw a complete angle by rotating the top shorter stick clockwise, making a full circle.
6. State that this is a complete angle from the word *completus* (Latin) meaning full.
7. Make a straight angle by placing the pencil in the next hole (so the semi-circle is smaller). Stop halfway.
8. State that this is a straight angle.
9. Allow each child to try.
Exercise Two: Right, Acute and Obtuse

Materials:
• Geometry sticks box
• Cork board
• Paper
• Pencil

Presentation:
1. State that you are going to make a new angle. (if continuing, turn paper over and reattach sticks)
2. Place the measuring angle on the board and make a right angle.
3. State that this is a right angle, from rit (Middle English) meaning straight up.
4. Make an obtuse angle, introduce as such (obtuses meaning blunt) and note that it takes up more space than the right angle.
5. Make an acute angle, introduce (acutese meaning to sharpen), and note that it takes up less space.
6. Have the children label the angles. Ask how they knew what angle it is.

Additional Activities:
1. Give each child a pair of sticks to connect and a measuring angle. Call off angles for them to make.
2. Write labels and give to each to make. Have choose a label to make stating after each what they’ve made and how they know what they were.
3. Follow-Up: Children can look at polygons and see what angles they have; they can make lists or drawings of angles in the environment and/or booklets with drawings and labels.
4. Just like going around and measuring things in the environment, they might go around and determine angles in the environment.
Exercise Three: Application to Triangle Nomenclature

Materials:
- Geometry sticks box
- Paper
- Pencil

Presentation:
1. Give each child 9 sticks to make three triangles (an equilateral, isosceles and scalene triangle) and ask them to make a triangle. (Note: make sure to give them the right sizes.)
   - equilateral - 3 equal sticks
   - isosceles - only 2 sticks are equal, other is either longer or shorter
   - scalene - none of the sticks are equal
2. Discuss what kind of a triangle they made noting the different lengths of the sides.
3. Label the triangles.
4. Mix up the triangles and ask them to identify each one and how they knew what it was.
5. Give children sticks to make 3 types of isosceles triangles.
6. Analyze the triangles by using the measuring angle over the angles of the triangle. Determine if it’s a right, obtuse or acute angle:
   - **Isosceles obtuse triangle**: one angle is bigger than a right angle and the two other angles are smaller than the other angles.
   - **Acute triangle**: all angles are acute.
   - **Acute equilateral triangle**: all sides are equal in length and all angles are acute.
   - **Acute isosceles triangle**: two sides are equal in length and all angles are acute.
7. Now or later depending on children’s interest, pass out sticks for children to make different scalene triangles (repeat the above activities).
   - **Acute scalene triangle**: three sides have different lengths and all angles are acute.
   - **Obtuse scalene triangle**: three sides are different and one angle is bigger than a right angle.

Additional Activities:
1. Make labels for the triangles and match them with the correct triangles.
2. Make posters, booklets, etc.

Notes:
- Special angles can be introduced after the children have been introduced to the 3 triangles.
- All triangles have two names, one tells the relative length of the sides and the other tells what kind of angle it is (e.g. scalene obtuse, equilateral acute, etc.)
Exercise Four: Nomenclature of Parts

Materials:
- Geometry sticks box
- Paper
- Pencil

Presentation:
1. Informally review what children know about angles. Have each pick two sticks to connect and make an angle.
2. Ask the children what they’ve made and how they knew what it was.
3. State that “today we will learn about the parts of the angle.”
4. Make an angle with the sticks.
5. State that this part (point to a side) and this part (point to the other side) makes an angle.
6. Name the parts of the angle: side, vertex and size or amplitude. Make a label for each.
7. Mix up the labels and have the children label the parts.
8. Put the sticks away and get the fraction divided circles. (curved sides are the amplitudes)
9. Have each child take one piece and label the various parts.
notes:
Exercise Five: Measurement of Angles

Materials:
• Montessori protractor
• Fraction insets (circles)

Presentation:
1. Present the Montessori protractor for measuring angles to the children.
2. State that “we can use this protractor to measure angles.” Discuss how 360° came to be from the Babylonians.
3. Demonstrate how to use the protractor with the fraction insets; line up the left side of the inset at zero. The size of the angle is the point of the other side.

Additional Activities:
1. Children can trace the divided circle angle on paper, color in the size, and write in the degrees.
2. Children can also use divided triangles measuring all the angles of a piece, coloring, and labeling. (degrees and type of angle)

Exercise Six: Addition and Subtraction of Angles

Materials:
• Montessori protractor
• Circle fraction inset

Presentation 1: Addition
1. Choose 2 circle fraction insets.
2. Trace the angles on a piece of paper.
3. Measure and record the degrees of each angle.
4. Place both of the angles in the Montessori protractor and add the angles.
5. Record the answer to the right of the drawn angles. Include operational signs.
6. Have children repeat with others.

Presentation 2: Subtraction
1. Start with a big angle.
2. Measure with the protractor and record the angle in problem form.
3. Choose an angle to subtract.
4. Measure and record the angle of the subtrahend.
5. Place the lower edge of the subtrahend piece at the degree point of the larger piece so that the smaller angle piece is enclosed where the larger piece was.
6. Look at the other side for the answer.
Presentation 1: Addition

120°

+ 72°

= 192°

Presentation 2: Subtraction

120°

- 72°

= 48°
Exercise Seven: Introduction of Circular Protractor

Materials:
- Ruler
- Circular protractor
- Paper and pencil

Presentation:
1. Draw an angle with a ruler; show how to position the protractor over the vertex with the line over zero.
2. Show how to read along the scale to determine how large the angle is; guild the child to using the correct set of numbers if two different ways are listed.
3. Invite the child to go back and measure other work they’ve done or they can make new angles.
4. Also, show them how they can make an angle of a certain size; draw a line, mark the vertex, and place the midpoint of the vertex on it.
5. Determine the degree on the protractor, mark and connect that point with the vertex to form an angle.
6. Children can also be shown an easy way to divide an angle in half (bisecting an angle).
7. Place the compass point on the vertex and make an arch; at the point where it intersects the legs, place the compass point and make two intersecting arcs above the first.
8. Mark a line from the vertex to the place where the arcs intersect.
9. Explain as an easier way than dividing the number or that it can be done with an angle that you don’t know the size of.
notes:
Exercise Eight: Two Angles Together

Prerequisite:
• Given after the presentation of the line

Materials:
• Geometry stick box
• Paper and pencil

Presentation: addition
1. Make three sets of two pairs of angles; place a green stick on top in one pair, and on the bottom in another (second stick should be one size smaller).
2. Place one side of one angle vertically and one side of the other angle horizontally to make a right angle.
3. State that “when two angles are placed next to each other and together they form a right angle, they are called complementary angles.
4. Move the vertical side of the angle down to a horizontal position.
5. State that when two angles make a straight angle, they are called supplementary angles.
6. With the third pair of angles, connect to make convergent lines; state as being called vertical angles.
7. Have children label; continue quizzing process as needed.
8. Take away one stick to form an adjacent angle. State “when two angles share a side, they are called adjacent angles.
9. Have children make a pair of adjacent angles; ask how they made each.
10. State that an adjacent supplementary angle is called a linear pair. Continue labeling and second period testing.

Additional Activities:
• Have the children label the angles.
• Children can draw the angles for independent work.
Complementary Angles
Supplementary Angles
Vertical Angles

Exercise 8
Sensorial Preparation for Theorems of Angles

Materials:
- Cork board
- Geometry sticks
- Crayons
- Multicolored pushpins

Exercise One: Alternate Interior and Exterior Angles
1. Set up the cork board with 2 parallel lines which run to the edge of the paper.
2. Introduce the interior and exterior space of the lines. Have the child color the interior and exterior space with a crayon.
3. Place a transversal line over the two parallel sticks. State that you have created a lot of angles.
4. Place two of the same colored pushpins on alternate sides of the transversal interior angle.
5. Introduce the angles the pins marked as alternate interior angles.
6. Ask the child to mark another pair of alternate interior angle.
7. Mark a pair of alternate exterior angles and ask the child what she thinks those are.
8. Introduce the angles the pins marked as alternate exterior angles.
9. Ask the child to mark another pair of alternate exterior angle.
10. Write labels for the angles.
11. Test the child; ask what angles are marked with the red pins, etc.
Exercise Two: Corresponding Angles Part I
1. Repeat the initial layout of the sticks above.
2. Place 2 same colored pins on the same side of transversal, one interior and one exterior.
3. Introduce as corresponding angles; discuss why it is called that.
4. Place one pin and have the child find it’s corresponding angle; ask how he knew that’s what it was.
5. Have the child find another pair of corresponding angles.
6. Combine with the concept of interior and exterior angles and do a three period lesson.

Exercise Threes: Corresponding Angles Part II
1. Prepare a paper with 2 parallel lines in the middle and color in the exterior and interior angles with different colors.
2. Draw a transversal line across the parallel lines.
3. Have the child mark the corresponding and alternate angles so that each equal angle is colored with the same x.
4. Cut out the lower left angle along its side, ripping at it’s third side. Superimpose it with other angles to find where it fits.
5. Cut also an interior angle and superimpose it with other angles to find where it fits.
6. Continue up the same side of the transversal matching each newly cut angle with ones that are still intact and those already cut out.
7. Match pairs of equal angles on the other side of the transversal observing that there are only two angles.
8. Children can repeat with their own piece of paper making their own lines and transversals.
Lines

Exercise One: Straight, Ray, and Segment

Materials:
- String
- Scissors
- Marker

Presentation:
1. Have the children feel the edge of the table and state that this is an example of a line.
2. Have the children look around the class and find other examples of a line.
3. While the children are looking around, place the string rolled up in both hands.
4. Slowly pull the string out in front of the children stating that straight lines don't have any ends.
5. Also show that lines can go in any direction in space by moving the string in different directions.
6. Hand one child a marker and another a pair of scissors; have the first child mark a spot on the line (string) and the other cut with scissors on the spot.
7. State that when you have a line with one end point, it is called a ray. The rest goes on forever.
8. Get more string and have the 2 children mark and cut simultaneously to form a segment.
9. State that when you have a line with 2 end points, it is called a line segment. (ends can be marked for better visibility)
10. Hand out string and have children take turns showing a line with no end points, a ray, and a line segment.

Exercise Two: Line Position

Materials:
- String with weight (water plumb)
- Geometry sticks
- Cork board
- Jar of colored water

Presentation:
1. Hold the string with weight and drop the weight to let it swing.
2. Label the position where it stops as vertical.
3. State that when the line goes up and down it is called a vertical line.
4. Shake the jar with colored water and have the children watch for it to settle.
5. State that this is a horizontal line.
6. State that lines that are not vertical or horizontal are called oblique lines.
7. Pass out string and have the children make different lines.
8. Place 3 types of line segments on the cork board using the geometry sticks and label.
9. Place a zigzag line on the board and introduce as a broken line. (label)
10. Mix labels, have place and explain how each knew what theirs was.
Exercise Three: Position of Two Straight Lines

Materials:
- Geometry sticks
- Cork board
- Three pairs of cut-out children (Pair = 1 boy + 1 girl): one pair neutral, one sad, one happy

Presentation:
1. Place three sets of 2 lines on the board: one parallel, one convergent, one divergent.
2. Introduce and label.
3. Introduce pairs of children. Place one of each of the neutral children on each of the parallel lines stating that they have never met, and will never meet and aren’t happy or sad about it.
4. Place sad children on the divergent lines moving away from one another stating that they’re sad because they won’t meet again.
5. Place happy children on the convergent lines facing the converging point explaining they’re happy because they’re going to meet.
6. Collect the labels and illustrated children; give each living child a label and have place with its corresponding pair of children in the appropriate location.
7. Continue testing procedure as needed.

Exercise Four: Intersecting Lines

Materials:
- Geometry sticks
- Cork board

Presentation:
1. Set up the board with 2 sets of intersecting lines.
2. Explain that there are two ways lines intersect; some do so in a way all angles formed are equal, all are 90 degrees. These are called perpendicular lines.
3. Others intersect unevenly, where 2 obtuse and 2 acute angles are formed. These are called oblique lines.
4. Label, etc.
Excercise Three: Position of Two Straight Lines

- **Parallel**: Lines that never meet, no matter how far they are extended. 
- **Convergent**: Lines that meet at a point. 
- **Divergent**: Lines that move away from each other. 

Images depict various positions of lines to illustrate these concepts.
IV Equivalence Continued

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Equivalence with the Pythagorean Theorem

Relationship of Squares and Triangles

Material:
• First Pythagorean plate I

Presentation:
1. Do a review by removing the white triangle and asking the children what kind of triangle it is. (right isosceles)
2. Replace the triangle and point to the squares; ask what those are.
3. Tell the children, “Today we are going to learn the relationship between the squares and triangles.”
4. Point out the triangle’s legs and state that the squares adjacent to the legs have the same length.
5. Ask what part of the triangle has the same length as the large square. (Hypotenuse)
6. Point to the hypotenuse and state that the hypotenuse is the same length as the Red Square.
7. Exchange the insets from one inset to the other to show their equivalencies.
8. State that there are more places where we can find equivalencies.
9. Remove the whole red inset and encourage the children to see that it is equivalent to the yellow and blue squares combined.
10. Note other equivalencies: yellow = 1/2 red, blue = 1/2 red.
11. After experience, give the rule: In the right angle triangle, the sum of the square built on the legs is equal to the square constructed on the hypotenuse.

Follow Up Activities:
• Tracing
• Relate to the Egyptians
Introduction to the Right Scalene Triangle

Material:
- Pythagorean Plate II or numerical plate

Presentation:
1. Remove the white triangle and ask the children what kind of triangle it is. (Right scalene triangle)
2. Tell the children, “Here we have another demonstration of the theory of Pythagoras.”
3. Explain, “In this frame the squares are all divided up. We can use it to verify that the square of the legs are equal to the square of the hypotenuse.”
4. Ask a child to help, place all of the small red and yellow squares into the frame of the large red square and place the small red squares into the frame of the blue and yellow squares.
5. Ask the children what they found; ask if it can be written out and do so:
   \[3^2 = 9\]
   \[4^2 = 16\]
   \[5^2 = 25\]
6. Children can do word problems, e.g. give the dimension of the hypotenuse and one leg, have them find the length of the other leg.

Note:
- Pythagorean triples: 3,4,5 5,12,13 7,24,25
- wrongly credited to Pythagoras
- Known & widely used in India, Babylon, Egypt & China
# Proving the Theorem with the Constructive Triangles

**Materials:**
- Triangular box
- Small and large hexagonal box
- 3 extra triangles in yellow the same size as the equilateral triangle in the small hexagonal box

**Presentation:**
1. Take out the **green right angle scalene triangle** from the triangular box and ask a child what kind of triangle it is.
2. Have the child name the **legs** and **hypotenuse** on the triangle.
3. Ask about the relationship of the legs and hypotenuse. (the sum of the square of the legs = sq. of the hypot.)
4. Tell the children, “I wonder if there is a relationship between other shapes built on the legs of the triangle.”
5. Place an **equilateral triangle** along each side of the scalene triangle:
   - **short leg** = 1 small red equilateral
   - **long leg** = 1 large yellow equilateral
   - **hypot.** = 1 gray equilateral.
6. Make 2 **rhombi** out of 2 **red equilateral triangles** and 2 **obtuse isosceles triangles**. Show the children how these are equivalent. (rhombus: a parallelogram with opposite equal acute angles, opposite equal obtuse angles, and four equal sides.)
7. Substitute the whole triangles with small equilateral and isosceles triangles to prove that it’s equivalent
   - **short leg** = 1 small red equilateral
   - **long leg** = 3 red obtuse isosceles
   - **hypot.** = 4 small gray equilateral.
8. Count the triangles to see if the sum of two legs equals the hypotenuse. ($1^2 + 3^2 = 4^2$)
9. Make other equivalencies by doubling each set of triangles creating rhombi; check to see if the equivalence is still true.
10. Repeat adding a third set to each; ask what you have now (trapezoids).
11. Explain this can also be done with hexagonals; lay out using extra yellow triangles.
12. Children can count equilateral small triangles as done previous noting that where three are used for a large triangle, they represent four.
13. Or children with a good command of fractions can consider the large triangle as the unit and determine fractionally how many small triangles are in the large:
Proving the Pythagorean Theorem with the Constructive Triangles

and this rhombus is equivalent to a rhombus made from two of those obtuse isosceles triangles.
Pythagorean III Plate

Materials:
- Pythagorean Plate III

Presentation:
1. Remove the red rectangles from the frame.
2. Slide the white triangle down and place the yellow and blue parallelogram in the frame. Note that the parallelograms are equal to the red triangles. Replace pieces.
3. Remove the yellow square, slide up the white triangle and put the yellow parallelogram in the space. (yellow parallelogram = yellow square) Replace pieces.
4. Remove the blue square and slide the white triangle up and fit the blue parallelogram in the space. (blue parallelogram = blue square) Replace pieces.
5. State, “We know the sides of a figure with equal heights and base are equivalent.”
6. Show this by placing the yellow square next to the yellow parallelogram in its rectangular slot. Compare the base and height in the frame.
7. State that “Since the yellow figures are equivalent and the yellow parallelogram is equal to the red rectangle, what can be said about the yellow square and red rectangle?” (they’re equal)
8. Show that the blue square is equivalent to the blue parallelogram.
9. State that since the blue square and parallelogram are equal, what can we say about the blue square and small red rectangle? Note that they are both equal.
Pythagorean Plate III

A

B

C

D

E
V Polygons & Circles

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Polygon and Circle Nomenclature

Nomenclature of Polygons

Materials:
- Geometry sticks box
- Plumb line (for presentation 2)

Presentation 1: Introduction
1. Make an irregular pentagon with the geometry sticks and ask the children what shape you are making and how they knew. (irregular polygon - not all the same sides and angles.)
2. Have the children label the different parts of the polygon, i.e. sides, consecutive sides, vertices, diagonals, angles.
3. Ask the children, “What are all the sides together called? Perimeter. And when we want to know how much is inside we call it area.”
4. Invite the children to choose sticks from the shorter ones to make and label their own polygons.
5. Read off the labels and have children identify these parts on their polygons.
6. State that you can do something else with the polygon. Place long sticks across the polygon from each non-adjacent angle.
7. Label these as diagonals; ask children how many fit on your polygon.
8. Invite children to place diagonal sticks on their own polygons; note that they can fit as many as there are sides.
How many diagonals in this polygon (hexagon)?
Presentation 2: Specific Terminology

1. Place an equilateral triangle made with the geometry sticks on a stand and hold a plum line from its top vertex.
2. Ask the children what kind of line it makes? (straight)
3. State that the line it makes from the top to the bottom of the triangle marks the altitude or height of the triangle.
4. Rotate the triangle and find the height; also do other triangles as well.
5. Make a right triangle and hold a plum line from its top vertex.
6. Ask the children where the height of this triangle is.
7. Make a scalene obtuse triangle and hold a plum line from its top vertex.
8. Explain that on triangles, the height is going to be in different places. The height of the obtuse scalene triangle (can be) external of the triangle.
9. Note that if you were going to draw the height of this triangle, you’d have to draw a dotted line off of the top vertex downward.
10. Introduce the Apothem. This is a line from the center to the midpoint of one of the side.
Presentation 3: Difference Between Squares and Rhombi

1. Make two regular quadrilaterals out of four of the same colored sticks; shape one into a square and the other as a rhombus.
2. Have the children name the quadrilaterals.
3. Place a measuring angle in one of the corners of the square; note that the angles are all equal.
4. Ask if any of the angles on the rhombus are equal.
5. Put diagonals on the figures and ask the children what they notice.
6. State that **the diagonals of the square are always equal while the diagonals of the rhombus are not**: one is always longer than the other. The longer diagonal is called the **major diagonal** and the shorter is called the **minor diagonal**.

---

**Square** and **Rhombus**

- **Square**: a square and rhombus made from eight equal sides. The square has 4 equal angles while the rhombus has two and two.
- **Rhombus**: notice the diagonals of the two different shapes.
Sum of Angles of Plane Figures

Materials:
- One envelope each of paper triangles, quadrilaterals, and polygons (regular and irregular), each with their angles similarly marked.

Presentation:
1. Choose a triangle; state that you’re going to try to find out the sum of its angles.
2. Children may give suggestions for how you could do this; explain that you don’t feel like doing all that.
3. Rip each angle off and match these together to make a 180° degree angle.
4. Invite children to try with their own triangles.
5. Ask what would happen with a four sided figure.
6. Use the same procedure with a quadrilateral putting the angles together to make a complete angle.
7. Repeat with the polygons.
8. After practice, give the rule: The sum of the angles of a figure is equal to the (number of sides minus 2) x 180°.
Nomenclature of the Circle

Materials:
- Cut out circumferences of circles corresponding with the geometric sticks.
- Divided circles (the whole, halves, and 6ths)
- Circle frame with an inscribed circle
- Cork board

Presentation:
1. Lay out one circle on a cork board with a geometric stick marking the radius.
2. Have the children identify different parts of the circle: circumference, center, radius, area, etc.
3. Place another geometric stick on the board at the end of the first stick making a line across the circle.
4. Ask the children how they would describe the line; ask them how many places it touches the circumference.
5. Children can draw this if they like and/or refer to the classified nomenclature.
6. Bring out the circular insets; illustrate the semi-circle and semi-circumference with the half circle inset.
7. Demonstrate an arch as part of the circumference marked off with 2 end points using the 1/6th circular inset piece.
8. Use this piece to also show a circular sector as the center part of a circle bordered by 2 radii, the arch being the center point.
9. Further illustrate this with a triangle inscribed in a circular inset.
10. Children should work with these names.
Nomenclature of Circle: steps 6-9

- semi-circumference
- semi-circle
- arch
- two radii of a circle
- circular sector
- an arch
Relationship Between a Line and a Circle

Materials:
- Cut out circle circumference
- Geometric stick box
- Cork board

Presentation:
1. State that we also have names for the relationship between lines and circles.
2. Place a circle on the board with a stick touching it at the edge.
3. State that when a line touches a circle at one point it is called tangent.
4. Place the stick across the circle stating when it touches two points it's called secant.
5. Place the line away from the circle stating when the line does not touch the circle at all it is called external.
6. Do a three period lesson.
7. State that you can also explore the distance between the center of the circle and a line in terms of the radius of the circle.
8. Note that this distance can be greater than, less than or the same as the length of the radius.
9. Place a tangent line on the circle; ask whether the distance is greater than, less than or the same as the length of the radius. (equal)
10. Repeat with the line placed secant to the circle (distance is less than the radius) and with the line placed externally (distance is greater).
Relationship Between Two Circles

Materials:
• Two cut out circle circumferences
• Geometric sticks

Presentation:
1. Place two circles so they are tangent to each other, that is, they meet at one point.
2. Note that there are two ways to do this; when the circles touch on the outside they are called \textit{externally tangent}. When one circle is inside the other and they touch at one point they are called \textit{internally tangent}.
3. Place the circles overlapping and ask the children what it is called (\textit{secant}).
4. Place the circles so they are not touching and state that they are \textit{external}.
5. Place a radius in the middle of the each circle; state you will explore the distance of the centers of the circles in terms of the radius.
6. Note when they are placed \textit{externally}, the distance between the centers is greater than the sum of the radius.
7. Arrange the circles \textit{internally tangent}, noting that the distance between centers is less than the sum of the radius.
8. Arrange the circles \textit{externally tangent}, noting that the distance between the centers is equal to the sum of the radius.
9. Arrange the circles \textit{secant}, noting that the distance between the center is less than the sum of the radius.
10. Inscribe one circle inside the other, note the centers are the same, and that this is called \textit{concentric circular sector}. 
externally tangent

internally tangent

secant
from the Latin: secare
meaning "to cut"

concentric circular sector
VI Study of Area

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Equivalence with the Metal Insets

Exercise One: Triangles

Materials:
- Two sets of Triangle Plates

Presentation:
1. Start with the first plate; point and ask what the shapes are called. (triangle & rectangle)
2. Tell the children, “I think the triangle and rectangle are equivalent but let’s exchange the pieces to make sure.” (Exchange pieces so the triangle is now filled out.)
3. Ask, “Where is the base on this triangle? The height?” Show the height by removing the trapezoid from the plate and sliding the top triangle on the right down.
4. Bring out the three plates in the second set and have the children name the figures.
5. State “I wonder if these are equivalent.”
6. Exchange around the insets of the second plates to show that the four figures are equivalent.
7. Refer to the first plate and state, “We know that the triangle and rectangle are equivalent, but how do we know? What can we say about the length of the base of the triangle and rectangle?” (The bases are equivalent.)
8. Then ask, “What can we say about the height of the triangle and rectangle?” (The height of the rectangle is half the height of the triangle.)
9. Summarize what’s been found: A triangle and rectangle are equivalent when their bases are equal and when the height of the triangle is twice that of the rectangle.
Exercise Two: Rhombi

Triangles with equal heights and bases are equal.

Materials:
• Plate with 2 rhombi, square, and an empty rectangle.

Presentation:
1. Bring out the plate with the two rhombi and have the children name the figures.
2. Have the children exchange the insets of all the plates to see if they are equivalent.
3. Have the children return the insets to their original frames.
4. Ask the children to identify the bases of the small rectangle and each rhombi.
5. Ask what can they say about the base of each rhombus and the small rectangle. (They are all equal.)
6. Ask the children what can they say about the height of each rhombus and the small rectangle. (They are all equal.) Note that the figures all have the same base and height.
7. State that you will show them something else; take one piece from each figure in the plate (each rhombus?) and place it in the rectangle below.
8. Ask the children if the triangles in the large rectangle are equivalent. (yes)
9. Ask the children what they know about the equivalencies of the triangles. (The heights and bases are equal therefore triangles with equal heights and bases are equal.)
10. Children can do their own work drawing equivalencies, etc.
Exercise Three: Trapezoids

Material:
- Trapezoid Plate

Presentation:
1. Ask the children to name the figures in the trapezoid plate.
2. Ask how many bases there are. State that in a trapezoid there is a *major* base and a *minor* base.
3. Tell the children, “Let’s see if the trapezoids are equivalent.” Exchange the trapezoids to show that they are equivalent.
4. Have the children place the divided trapezoid in the rectangular space on the plate to establish that the trapezoid and rectangle are equivalent.
5. Have the children think about the relationship of the bases and heights of the trapezoid and rectangle.
6. Ask, “What are the bases of the trapezoid equal to?” (The major and minor bases of the trapezoid equal the base of the rectangle.)
7. Ask “What about the height of the figures?” (The height of the rectangle is 1/2 the height of the trapezoid.) Show this by placing the trapezoid triangle on the right below the triangle on the left.
8. Ask “What does this tell us about the relationship between the two?”
9. Summarize: The trapezoid and rectangle are equal if the base of the rectangle equals the major and minor bases of the trapezoid together, and the height of the rectangle is 1/2 that of the trapezoid.
Exercise Four: Decagons

Material:
- Decagon plates
- Partially divided long rectangular plates

Presentation:
1. Bring out the decagon plates; ask the children to name the polygon.
2. Have the children exchange the pieces of the decagon to see if they are equal.
3. Introduce the rectangular plates and have the children check if the solid part of the rectangle is equal to the divided parts by placing the whole rectangle over the divided segments on each plate.
4. Tell the children, “I wonder if the rectangle is equivalent to the decagon. What do you think? Are they equal?”
5. Place the divided pieces of the divided rectangle into the frame of the decagon; note that the decagon is equal to the rectangle.
6. State, “So we know that the solid rectangles are equal, the decagons are equal and the divided rectangle is equal to the decagons.”
7. Ask, “What about the whole rectangle?” Determine that it is equivalent to the decagon as well.
8. Review what's been proven; all figures are equivalent.
9. Tell the children, “Now that we know that all the figures are equivalent, we need to find the important lines of these figures, the heights and the bases.”
10. Point out the perimeter and the apothem on the decagon.
11. Explain that on the decagon, the “base” (use this term only to help see analogy with area of triangle theorem) is its perimeter and the “height” is its apothem – the distance from the center to the middle of one side.
12. Take out the short, whole rectangle and hold its base below the rectangle (divided) next to it, noting that this is worth 1/2 of the decagon’s perimeter (base = 1/2 perimeter).
13. Ask a child, “How much of the perimeter is the base equal to?”
14. State, “Let’s look at the heights. The height of the rectangle is equal to the apothem, or the height of the decagon.” Show this by taking a small piece of the rectangle and superimposing it to the height or apothem of the decagon. (center and midpoint base of the decagon)
15. State further, “So we can say the decagon and the rectangle are equivalent when the base of the rectangle is equal to 1/2 the perimeter and the height is equal to the apothem.
16. State that we also know that the decagon is equal to the long rectangle (bottom half of inset); have the children find the relationship.
17. Note that the base of the long rectangle is equivalent to the perimeter of the decagon and the height is equal to 1/2 of the apothem.
Place the whole rectangle over the divided segments to see if they are equivalent.

The pieces of this divided rectangle fills the place of the decagon in the frame.

So, the rectangle and decagon are equivalent. (Same can be show with other rectangle and that both rectangles are equivalent.)
Area of Plane Figures

Exercise One: Rectangles

Materials:
• Yellow Area Materials: four rectangular pieces lined up vertically with the mostly blank rectangle placed on the left, the two rectangles with one set of lines in the middle, and the rectangle with a grid placed on the right.

Presentation:
1. Point to the almost blank rectangle and tell the children, “I’d like to find how big the surface of this area is.”
2. Place the vertically striped rectangle over the first rectangle, point to vertical marks on the first rectangle and state, “I could extend these lines and I would have some stripes, but I still wouldn’t know the surface area of the rectangle.”
3. Place the horizontally striped rectangle on top, point to the horizontal marks on the first rectangle, and state “I can extend another set of lines, but I still wouldn’t know the surface area of the rectangle. What could we do?”
4. Show the plate with grids and state “What if we extend the lines in both directions? Now do we have a way to express what the area is?”
5. Ask the children how they think we could find the area with the grid plate.
6. State “We could count the squares but there is an easier way.”
7. Help the children figure out that you could count the sides and multiply the base and the height to get the surface area of the rectangle.
8. Children can do work on graph paper by drawing squares and rectangles and calculating their areas.
9. Have the children work on this until they solidify the concept that to compute the area of a rectangle, multiply the base and the height.
Exercise Two: Parallelogram

Materials:
- Whole and divided parallelograms

Presentation:
1. Do a short review on how to find the surface area of a rectangle.
2. Show the whole plate of the parallelogram and ask the children, “What if we wanted to find the area of this parallelogram?”
3. Bring out the divided parallelogram plate. State that they are the same.
4. Move the smaller piece to the other side of the parallelogram (forming a rectangle); ask if it is equivalent to the other piece.
5. State that the figures are still equivalent because we didn’t add anything, we just moved it around.
6. State that when we calculated for the area of the rectangle, we multiplied the base and the height. Ask if you can do the same with the parallelogram and point out the dimensions to multiply.
7. Go on to the next exercise in the same lesson.
Exercise Three: Triangles  
Part I: Acute Isosceles Triangles  

Materials:  
- One whole and one divided acute isosceles triangles  

Presentation:  
1. Bring out the whole triangle and have the children name the figure.  
2. Ask what if you wanted to find the area of the whole acute isosceles triangle, what would you do?  
3. State that you have something to help them find the area. Bring out the triangle divided in the middle.  
4. Place the half triangles around the whole triangle to form a square.  
5. Have the students figure the area of the triangle noting that you need to compute the area of the square and divide it by two.  
6. Ask the children, “What if you just have the divided triangle, how can we compute for the area?”  
7. Have the students watch while you transform the divided triangle to a rectangle.  
8. State that to compute the area, you have height and half of the base.  
9. Present the third triangle and have the children move around the pieces of the triangle to compute for the area.  
10. State that you have the base and half of the height.
Part II: Right Isosceles Triangle

Materials:
- Whole and divided right isosceles triangles

Presentation:
1. Present the whole triangle and have the children name the figure.
2. Present the divided triangle and ask what can be done to show the area of the triangle.
3. Put the pieces together to make a square; note what you did, e.g. multiplied the height by the base divided by 2.
4. Ask if there is another way to move the pieces around to form a rectangle; e.g. the height divided by 2 times the base.
5. Explain that both of the formulas will yield the same surface area.
Part III: Obtuse Isosceles Triangle

Materials:
- Whole and divided obtuse isosceles triangles

Presentation:
1. Bring out the whole triangle and have the children name the figure.
2. Put the whole and divided triangles together and note that they don't form a square.
3. Find the base and height (external) and multiply, but note that you only want 1/2 of this and divide by 2.
4. Isolate the divided triangle. Ask where the height and base are.
5. Move the top piece away and ask what you took away. (half of the height); determine that the area is: Area = b h/2
6. Another way to illustrate this is to move the divided triangle around and form a rectangle.
Deriving the Formula with the Yellow Material

Materials:
- Yellow area material
- Lower case letters a, p, b, d, and h
- Capital letters A, B and D
- + sign, brackets, a 2, and + sign, black dividing line

Presentation:
1. Present the whole yellow rectangle and review how to compute the area of the rectangle.
2. Have the children mark the parts of the rectangle as small b for base and small h for height.
3. Place the two letters together under the rectangle noting that these are multiplied.
4. Present the whole parallelogram and have the children mark the base and height with letters then place the formula under the parallelogram.
5. Next, go on to the triangle and note that there are several ways to compute for the area.
6. Go through each triangle figure with the children, having them mark the base and height with letters, then form the formula underneath the figure.
deriving formula with area material
Iron Material Formulas

Materials:

- Metal insets: triangle, trapezoid, rhombus, decagon and pentagon
- Area formula letters and symbols

Exercise One: Triangle and Trapezoid

1. Present the triangle frame and have a child pick out the elements of the formula needed for writing its area.

2. Place the triangle pieces in the rectangle frame; ask, “What formula do we use to compute when using this rectangle?” (bh-rectangle; bh/2-triangle)

3. Present the trapezoid insets and take out the elements needed for writing the formula.

4. Identify and note the height and major and minor bases of the trapezoid.

5. Move the pieces of the trapezoid into the rectangle frame.

6. Note the base of the rectangle is equal to the major and minor base of the trapezoid.

7. Mark the major base as capital B and the minor base as small b.

8. Put the elements together arriving at a formula.

9. Area of Trapezoid = (B + b) h/2 [major and minor bases times half of its height.]
Exercise Two: Rhombus

1. Bring out the 3 rhombi frames; one containing a rhombus divided along its major diagonal and the other on the minor diagonal, and the other with a solid rhombus and rectangle divided to form a rhombus.

2. Make a rhombus using half the rhombus divided along its minor diagonal and the two right triangle pieces from the divided square.

3. Have a child mark the rhombi using a capital D for the major diagonal and a small d for the minor diagonal.

4. Make the pieces into a rectangle and ask the children how to find its area.

5. Ask “Which part of the rectangle is the base?” (minor diagonal; have the children mark it with a small d).

6. Have the children identify the height of the rectangle (1/2 the major diagonal) and mark it with a capital D/2.

7. Put the elements together and arrive at the formula: \( d \cdot \frac{D}{2} \)

8. Make a second rhombus replacing the minor diagonal piece with a major diagonal piece.

9. Note the minor and major diagonals; make into a rectangle and ask which part of the original is the base (D).

10. Ask what the height is (minor diagonal divided by 2); lay out as: \( D \cdot \frac{d}{2} \)

\( d = \text{minor diagonal} \)

\( D = \text{major diagonal} \)

\( \text{base} = D \)
Exercise Three: Decagon

1. Do a review; present the plate with the short rectangle and ask where the apothem and the perimeter of the decagon are on the plate.

2. Refer to the base of the short rectangle and ask what part of the decagon it's equal to [one-half of the decagon's perimeter]. Label it as \( \frac{p}{2} \) for 1/2 of the perimeter.

3. Determine that the height of the short rectangle is equal to the apothem of the decagon. Place one small triangle from the rectangle on the decagon to show this.

4. State that one formula for computing the area of a regular polygon is: \( A = \frac{a}{2} \) [where \( a \) = apothem.]

5. Bring out the plate with the long rectangle. Ask if it is equivalent to the decagon.

6. Refer to the base of the long rectangle and ask what part of the decagon it's equal to; label it as \( p \) for perimeter.

7. Ask what the height is; note as 1/2 of the apothem.

8. Give the second formula for surface area of a regular polygon. \( A = \frac{p}{2} \)

9. State that there is a third way to arrange the formula to give the area of a decagon.

10. Point out that if we combine the two rectangles together, it will be equal to two decagons. The formula would be: \( \frac{ap}{2} \)
Exercise Four: Pentagon

1. Introduce the pentagon frames; one whole and one divided into 5 equal parts.
2. Show the equivalencies of the pentagons.
3. State that you can make something rectangular-like with the pieces. Lay out the pieces making a trapezoid.
4. Trace the pieces on paper, making a trapezoid.
5. Explain that you want to get it into a rectangle.
6. Cut off one of the end pieces and move it to the other side to make a rectangle.
7. Note that the base of the rectangle is equal to half of the perimeter of the pentagon.
8. Show that the height is equal to the apothem by placing the cut off piece in the pentagon frame.
9. Set out the formula for the pentagon: \( \text{Area} = \frac{a \ p}{2} \)
Exercise One: Introduction to the Circle

Introduction:
This lesson helps children discover that the circle is a many sided polygon and leads them to discover \( \pi \).

Materials:
- Geometric Cabinet Polygon drawer
- Circles that have been constructed on paper

Presentation:
1. Take out the inset and frame for the largest circle from the geometric cabinet.
2. State that the circle is a many sided polygon and remove the circle from its frame.
3. Beginning with the pentagon, place polygons in order from least to the most number of sides into the circular frame and ask what’s happening to the circular segments, i.e. the space between the each polygons sides and the circular frame.
4. Note that as the number of sides on the polygon increases with each new one the space created gets smaller.
5. Present a hexagon to the children and ask how they would find its perimeter. (Measure a side and multiply it by the number of sides)
6. Ask the child to find the circumference of a circle. (They can’t because they don’t know the length of the side.)
7. Draw a straight line on a piece of paper, marking an endpoint.
8. Show the children how to measure the circumference using the line by marking a point on the circle, matching it with the beginning point of the line, rolling the circle and marking where the beginning mark comes up again.
9. Mark off the line in diameters of the circle. Ask the children how many circles fit. (3 and a bit)
10. Let the children repeat with other circles of different sizes, lining up the beginning points on the paper on top of one another.

Notes:
- Relate that the Greeks found a relationship between the diameter and the circumference of a circle; they called it \( \pi \), after one of their letters noting it had a value of 3 and some.
- Note that the Mathematicians couldn’t find an even number for this but have determined it to be 3.14.
- Show the formula for this: \( C = d\pi \)
- Have the children find the formula with just the radius: \( C = 2r \)
notice the area diminishing as the number of sides grows.

steps 7-10

mark a point on the circle.

one complete revolution

this distance on the line

is equal to the perimeter of the circle.

mark the endpoint of the straight line.

mark the endpoint of the circle.
Exercise Two: Finding the Area of a Circle

Materials:
- An envelope with two different colored circles of the same size
- Two rectangles that are equivalent to the circles and another that is twice the area.
- Two additional circles equivalent to the above circles: both are divided into 10 equal parts, but the 10th part on one is divided into half.
- Fraction lines that are both long and short
- Area symbols, i.e. A, 2, c (circumference), d (diameter), r (radius), π(pi) with 3.14 printed on the reverse.

Presentation:
1. Set out the material; discuss the parts of the circle and verify the equivalence of the circles.
2. Ask the children how they would find the area of a rectangle; state that you’d like to find the relationship of the rectangle to the circle.
3. State that you’ll take the circle apart and make something that looks like the rectangle.
4. Align the arcs of the circle divided into 10 along what would be the base; note that this doesn’t work as is, and fill in with the pieces from the other divided circle.
5. Compare the constructed rectangle to the regular rectangle; point to the arcs and ask what part of the circle the base is (circumference).
6. Ask what part of the circle the height is (radius); form the formula: \( A = cr \)
7. Note that the rectangle is not the area of one circle, but of two; change the formula to: \( A = cr/2 \)
8. Ask the children how to find the circumference of a circle; establish as: \( c = d\pi \)
9. Note that if you don’t know the diameter, you can use: \( 2\pi r/2 \) which can be cancelled to \( A = \pi r \) or \( A = \pi r^2 \)
10. Compare the split rectangle and the circle.
The Relationship Between the Apothem and The Side of a Regular Polygon

Materials:
- Two or three regular hexagons, one with 1 cm sides, 1 with 1 dm sides, and 1 with 1 m sides.
- A measuring stick

Presentation:
1. Propose the question: If a circle is a regular polygon and there’s a special relationship between the circumference and the radius, is there a similar relationship between the perimeter and apothem of other regular polygons?
2. State let’s see what we can find out about the apothem and the base.
3. Have a child measure the base of the small hexagon, then the apothem (b = 1cm, ap = 0.8cm)
4. Repeat with the 1 dm hexagon ( b = 10 cm, ap = 8.6cm).
5. Note that the side is a bit bigger than the apothem in each case; note that this relationship is just for the hexagon.
6. Ask what this information can tell us: if we didn't know the apothem but knew the size of the side, we could multiply by 0.8 to get the apothem. (or we could divide if we had the apothem but not the side)
7. Children can find other relationships; they can compute any apothem using the chart (below) and the length of a side.

Special Fixed Relations of the Apothem to the Base:

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Apothem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>0.288</td>
</tr>
<tr>
<td>Square</td>
<td>0.5</td>
</tr>
<tr>
<td>Pentagon</td>
<td>0.688</td>
</tr>
<tr>
<td>Hexagon</td>
<td>0.866</td>
</tr>
<tr>
<td>Heptagon</td>
<td>1.038</td>
</tr>
<tr>
<td>Octagon</td>
<td>1.207</td>
</tr>
<tr>
<td>Nonagon</td>
<td>1.374</td>
</tr>
<tr>
<td>Decagon</td>
<td>1.538</td>
</tr>
</tbody>
</table>

for a hexagon:
length of side multiplied by .866 equals the apothem

side= 1cm

apothem = 8.6cm

NOT to scale

side = 1dm (= 10cm)
VII  Solid Geometry

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Presentation of Volume

Introduction:
This lesson helps children to become aware that there is space in the interior of solid figures.

Materials:
• The Pink Cubes from primary
• Box of 250 2 cm. Cubes

Presentation:
1. Present a cube from the pink tower and ask what shape it has.
2. State that you are going to try to build it with the small cubes.
3. Lay out the 3 dimensions of the cube holding the corresponding edge on the pink cube next to each to make sure each side is the correct length.
4. Fill in the base first then build the subsequent layers up one at a time.
5. State that you wonder how many small cubes make the solid one.
6. Have the children decide how they are going to figure this out. \[3 \times 3 \times 3 = 27\]
7. State, “Let’s see what solids we can make”; have each child choose a solid (e.g. pink cube or some other sensorial material), build it with the small cubes, and calculate its volume.
Building Solids with the Same Amount of Cubes

Materials:
• Box of 250 2 cm. Cubes

Presentation:
1. Ask a child to count out 24 cubes.
2. State that we are going to see how many different ways we can arrange these.
3. Do one figure and have the children experiment with others. Note that all the figures are equivalent.
4. Show the children how to write this out. Count the width, length and height and record, e.g. 1 x 2 x 12 or 2 x 2 x 6, etc.
5. Children can do the arithmetic for this, they should get 24 each time.
6. Do this with other even amounts.
Calculation of Volume

Introduction:
This lesson helps children to conceptually understand that to find volume, you deal with three dimensions, two of which make the base times the height.

Materials:
- Box of 250 2 cm. cubes
- gray rectangular prism
- 5 yellow rectangular prisms

Presentation:
1. Present the gray prism and have the children label it.
2. Say you’ll build another one like it out of the yellow prisms.
3. Stack the yellow prisms with the squared one on the bottom, followed by the prisms with stripes, the notched prism, and the blank one on top.
4. Ask if the gray prism and the yellow prisms are equivalent (they should be), and state “If you wanted to find out how much space is contained in the prism (volume) this one made of 5 pieces can help us.”
5. Note that you could build it with the small white cubes but there is an easier way.
6. Take the top prism off and ask the children if they have seen this before; continue to lay the prisms out on the table.
7. State that the one on the bottom (with grids) will help us find the area of the base; have the children do so.
8. State that we don’t want just the area of the base; restack the prisms counting while doing so
9. Note that we want to know what the whole thing is worth and there are 5 of these so we take the base 5 times.
10. Suggest they figure it out using the small white cubes and proceed to build the cube out in the dimension of the prisms (don’t fill in).
11. State that we can calculate how many times to take it from this; help the children to do so by multiplying the base times the height.
12. Set the gray prism on end and ask what if it was this way; children may want to lay out the cubes and calculate or may know that it’s the same.
The two rectangular prisms are equal in volume.
So is one made of 250 cm\(^3\) pieces.
Equivalence Between Prisms Having Various Bases

Materials:
- 4 large prisms one with a rhombus as a base
- an equivalent divided rhombus
- one with a hexagonal base
- and an equivalent one to this with its base divided

Presentation:
1. Present the rhombic prisms and ask a child its name or teach it through reason (a rhombus with height is called a rhomboid).
2. Position the 2 rhombic prisms to show they have the same base and height.
3. Ask a child if she would like to change the divided rhombic prism into a different shape (i.e. forming a rectangular prism).
4. Ask the child what they know about the 2 now (they are equivalent).
5. State that there is something else we can do, we can work with the halves of the divided one.
6. Have a child change the divided halves into a rectangular prism; ask if the prisms are equivalent and how the child knows they are.
7. The child may know a number of ways, may look at the base and remember it from their study of area work.
8. Repeat with the hexagonal prisms; determine that the bases and heights are the same and that they are equivalent.
9. Ask a child to see what else can be done with the divided hexagonal prism.
10. Have the child reposition it into a rectangular prism; note that it is equivalent to the whole hexagonal prism.
11. Position the edge pieces of the divided hexagonal to form a rhomboid and note that this and the remaining rectangular prism are equivalent to the hexagonal prism.
12. Talk through how to find the volume of these, i.e. area of the base times the height.
Equivalence Between Prisms Having Various Bases

same height

same base

change pieces around to make rectangular prism with same volume.

hexagonal prisms - for reference:
Derivation of the Volume Formula

Prerequisite:
Children have understood discussions on how to calculate volume and know that it is done in terms of cubes. They also know that the two pairs of prisms (rhomboid and hexagonal) are equivalent, i.e. one rhomboid is equivalent to the other, the solid hexagonal prism is equivalent to the divided one.

Materials:
- Tickets for the symbols: Ab (area of base), H (height), V (volume), =
- 4 prisms

Presentation:
1. Present the rhomboid; state that you found out how to get the volume of it by finding the area of the base and multiplying this by the height.
2. Layout the tickets:
   \[ V = Ab \cdot H \]
3. Have the child plug in the area of the base for the formula for the rhomboid.
   \[ V = D \cdot h \]
4. The child can then measure with a ruler and calculate.
5. As a check, the children can arrange the divided rhomboid into a rectangle and calculate:
   \[ B = \frac{D \cdot d}{2} \]
   \[ d = \text{minor diagonal} \]
   \[ D = \text{major diagonal} \]
6. Children can work with just the 2 half pieces of the rhombus formed into a rectangle; these should be equal to half the rhombus.
7. Plug in the hexagonal formula, finding the apothem from the table showing these:
   \[ V = ap \cdot H \]
   \[ a = \text{apothem} \]
   \[ p = \text{perimeter} \]
8. Children can compute values and check. They can continue work by doing problems with imaginary figures.
Solids of Rotation

Exercise One: Rotating Figures

Materials:

- A tray of fine sand
- A narrow rectangle
- A right angle triangle

Presentation:

1. State, “Today we are going to find something out about the rectangle.”
2. Have the children watch as you rotate the rectangle around in the sand tray making a circle.
3. Have the children try to imagine what figure would have a circle as the base and the rectangle as the height. (cylinder)
4. Allow each child to rotate the rectangle in the sand.
5. State, “Let’s look at a different shape.”
6. Repeat with the triangle; note that a circle is formed again.
7. State that this time with a solid on it you wouldn’t have a cylinder, but what? (a cone).
Exercise Two: Finding the Volume

Materials:

- Hollow Solid – a prism, a pyramid with the same height and base and a shorter prism.
- A tray
- Bucket of sand
- Volume tickets with symbols

Presentation:

1. State that you will try to find the volume of these figures by applying what we know about calculating volume.
2. Ask how you would find it for the prism (H x B). Note that the pyramid and the prism have the same base and height, but aren’t the same.
3. State that you have to find the relationship between the prism and the pyramid, and that we can do so by measuring.
4. Have one child fill up the pyramid with sand, level it off and have her pour it into the prism to see how far it fills it up.
5. Repeat until the prism is full (3 times altogether).
6. State let’s find out how to find the volume of the pyramid.
7. Lay out the tickets, talking through:

\[ V = \frac{1}{3} Ab H \]

8. Introduce the small prism; note this is 1/3 of the large prism.
9. Have the children show this as true with the sand.
10. Derive the formula:

\[ V = \frac{H}{3} Ab \]

11. A child may note that the small prism and pyramid are equivalent.
Exercise Three: Finding the Volume of a Cone

Materials:
- Cone and cylinder pieces from the geometric solids
- Volume tickets with symbols

Presentation:
1. Hold up the cylinder piece; ask the children how they could find the volume of this figure.
2. Based on what the children know, determine the formula as: \( V = Ab \times H \)
3. Introduce the cone; show as having the same base and height as the cylinder.
4. Ask the children if they can find the formula for this figure; determine it to be: \( V = Abh/3 \)
5. Children can try this for themselves by making models and calculating their volume.
6. Using these concepts, children can explore the volumes of other solid figures.
7. Children who are interested can be lead into textbooks for further research.

\[
V = Ab \times H \quad V = Abh/3
\]
**Slant Height and Polyhedral**

**Materials:**
- A pyramid with an equilateral triangle as its base made out of yarn and straws
- A piece of paper the same size as the side of the pyramid with a line drawn in the center marking the height
- Plumb line

**Presentation:**
1. Hang the plumb line through the middle of the pyramid to mark the height and label it as such.
2. Note that the figure has three lateral planes that are congruent.
3. Place the piece of paper on one of the sides of the pyramid.
4. Label the line it marks as the slant height. (highest point to the middle of one of the base)
5. Present a polyhedron to the children (can be made by taping together 4 of the same equilateral triangles from the constructive triangles).
6. Label the polyhedron as such.
7. Children can make these by computing the area of the outside surface and cutting it out of cardboard.
three lateral planes that are congruent

Slant Height
Lateral and Total Area of Solids

Exercise One: Area of a Rectangular Prism

Materials:
- A long ruler
- News print
- Geometric solid figures

Presentation:
1. Present the rectangular prism.
2. State that you wonder how much paper it would take to cover the four surfaces of the prism.
3. Label this as the lateral area.
4. Draw a straight line on the paper.
5. Place one edge of the base of the prism along the line at its beginning point.
6. Draw the outline of the prism; roll it over and again draw its outline.
7. Repeat two more times so the surface area of the prism is represented on the paper.
8. State that you have a rectangle and know how to find the area with the formula b x h.
9. Ask how you got the base (by rolling the prism along the line).
10. Note that you made the perimeter.
11. State that you can multiply the perimeter by the height to get the lateral area (perimeter is the base, the height of the figure is the height).
12. Ask the child what we haven't drawn yet (the bases of the rectangular prism).
13. Draw these off of the first rectangle drawn.
14. Determine that these are 2 times the base squared.
15. Note that to get the total area you use the formula:

   Total Area = \( PH + 2Ab \)
Exercise Two: Area of a Triangular Prism

Materials:
- A long ruler, Newsprint, Triangular prism

Presentation:

1. Present the triangular prism, and ask the children how they would find the surface area of this figure.
2. Allow them to do so by drawing a line, placing one edge of the base along it and drawing the outline for each surface the same as for the rectangular prism.
3. Note that the base of it is the perimeter and the height is the height so the formula for the lateral surface is the same: \( PH \) (perimeter \( \times \) Height [with a capital \( H \) - different than \( h \) below.])
4. Draw the base of the prism to find the total area, and add 2 times the area of the base.
5. Ask what kind of base it is (a triangle).
6. Review that to find its area you would set it up as: 
   \[
   2 \times \text{bh} = \frac{bh}{2}
   \]
7. Determine the total area as being: Total Area = \( PH + bh \)
Exercise Three: Area of a Cylinder

Materials:
- A long ruler, Newsprint, Cylinder

Presentation:

1. Present the cylinder.
2. Help the children see that to find the area, they do it in the same manner by marking a starting point on the base and rolling it one time around on the line.
3. Mark the height at both ends of the line and draw 2 parallel lines to the base line then connect the 2 with a line to create a rectangle.
4. Ask what part of the cylinder is the base of the rectangle (the circumference).
5. Determine the lateral surface area as CH (circumference time height).
6. For the total area, mark the bases, determine that the area of the base is pi r^2 and derive the formula:
   \[
   \text{Total Area} = CH + 2\pi r^2
   \]
Exercise Four: Area of a Square Pyramid

Materials:
- A long ruler, News print, Square pyramid

Presentation:
1. First see if the children can figure how to calculate the surface area of the square pyramid on their own.
2. They should again draw a line, line up the edge of the base with it, and draw the outlines of the surface.
3. Have the children color these in at this point.
4. Ask if they have ever made a rectangular form like this before.
5. Finish it off as they have done before by doubling the area to form a rectangle.
6. Note that the base is equal to the perimeter, and the height is equal to the slant height. (P x sh)
7. Point out that if you do just that, you get lateral area for two pyramids, this should be divided by 2:
   \[
   \frac{P \times sh}{2}
   \]
8. For the total area, add the base to get:
   \[
   \text{Total Area} = \frac{P \times sh + A_b}{2}
   \]
Exercise Five: Area of a Cone

Materials:
- A long ruler
- News print
- Cone

Presentation:
1. This is an optional exercise.
2. Draw a line and place the point of the circle at the end.
3. Mark a point on the base and line it up with the line.
4. Roll the cone one time around drawing the arch produced while doing so.
5. Note that this is most similar to a triangle: the base equals the circumference, the height of the triangle is equal to the slant height.
6. Determine the lateral surface to be:
   \[
   \frac{C \cdot h}{2}
   \]
7. For the total area, draw in the circle and derive the formula as:
   \[
   \text{Total Area} = \frac{c \cdot h + \pi r^2}{2}
   \]