## Math I

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The Montessori Mathematics Curriculum

The Montessori mathematics curriculum is a comprehensive, multidimensional system for learning mathematical concepts based on the needs and developmental characteristics of preschool and elementary aged children. Designed as a means for assisting children in their total mental development, the method focuses on fostering depth of understanding and the discovery of mathematical ideas rather than the memorization of isolated facts and procedures. Instruction is individualized and manipulative materials, each isolating a single idea, are used to help lead the child from sensorial exploration to an abstract understanding of mathematical concepts. Students move through the curriculum at their own pace and are able to discover and correct their own mistakes through the built-in control of error incorporated in many of the mathematical material. A wide range of activities, differing in depth, complexity, and scope, are introduced to provide the child with repetition of basic skills in a variety of contexts.

The materials used in the Montessori mathematics curriculum present ideas in concrete form and are sequenced to gradually move the student toward abstract comprehension. A material itself may offer various levels of difficulty to help the child reach an internalized understanding of a concept or the child may be moved toward abstraction by working through a series of materials that become increasingly abstract. In addition to directly preparing the child for the next material in a series of exercises, each material also provides a foundation for further mathematical activities by incorporating concepts that are absorbed by the child at a sensorial level. For example, while the direct aim for a set of numerals cut from sandpaper is to introduce the symbols for the quantities from one to nine, the indirect aim is to physically prepare the child for writing the numerals at a later time. It is the accumulation of sensorial knowledge, as well as the child’s direct experience with sequential materials, that lays the foundation for an abstract understanding of mathematical concepts in the Montessori system of education.

The high degree of consistency found between the materials used in the Montessori mathematics curriculum also helps children in the process of attaining abstract understanding. The uniformity of size and color within the materials, such as all the bead material being made of the same sized beads and the consistent color coding of the hierarchy designations, facilitates the transition from one material to the next and adds an element of coherence to the curriculum as a whole.

The Montessori elementary mathematics curriculum builds directly on the work completed by the child in early childhood classrooms. At the preschool level, the child engages in self-construction and is in a critical or sensitive period for absorbing mathematical concepts. Basic mathematical ideas are presented in concrete form, and materials are introduced to individual or small groups of children depending on each child’s experience and level of maturity. In each group of work, the quantity is given first and is then followed by an introduction to the written symbol. The work culminates in an activity combining the two. As the child progresses through the curriculum, repeating activities at her own discretion, she is lead toward an un-
standing of basic mathematical processes and is prepared for the advanced mathematic and logical exercises she will encounter at the elementary level.

The study of mathematics in the elementary classroom continues to focus on the use of manipulative materials that lead the child toward abstraction, but differs according to the developmental characteristics of the 6 to 12 year-old child. Elementary aged children are socially oriented and enjoy using their ability to reason and to complete large, challenging projects. To cater to these characteristics, the Montessori mathematics curriculum encourages students to collaborate on activities and to derive for themselves the formula, algorithm, or rule necessary for abstractly completing a mathematical procedure. Many exercises allow for the possibility of extensive work while the flexible structure of classroom time enables students to complete large projects and investigations. Presentations are usually given to small groups of children and repetition at this level is provided through a variety of activities including those initiated by the students themselves.

Mathematical work at the elementary level is also set apart from the early childhood program by being viewed from an historical perspective. Throughout the Montessori system, a general impression of the whole is introduced before an analysis of the parts is undertaken. In the primary classroom, a view of the whole in mathematics is given by introducing the decimal system using concrete representations of the hierarchy of numbers before advanced counting and the operations of addition, subtraction, multiplication, and division are presented. At the elementary level, a vision of the whole in mathematics is obtained by placing mathematics in its historical context. The Story of Numbers, presented at the beginning of each school year, introduces the child to the major systems of counting that have evolved over time and how these systems developed to fulfill basic human needs. The relevance of mathematics in human society is emphasized in the story and a connection between the study of mathematics and other curriculum areas is established. Later presentations on the history of measurement and the history of geometry provide further details in relation to the whole and continue the child’s exploration of historical mathematics.

The study of geometry forms a separate curriculum area in the Montessori system of education. However, geometry concepts are introduced and examined in the same manner as general mathematical ideas and the two curriculums are explored concurrently throughout the school year. In both areas children are encouraged to initiate their own follow-up activities after presentations and to share their completed projects with others. Independent work in both areas helps the child to relate mathematics to the real world and provides opportunities for working with mathematical ideas in meaningful situations. Developing depth of understanding and the reasoning abilities of the child are the primary aims of both curriculum areas.

The following is a description of the four areas of study that collectively form the Montessori mathematics curriculum: (a) early childhood mathematics, (b) elementary mathematics, (c) early childhood geometry, and (d) elementary geometry.
The Early Childhood Mathematics Curriculum

The study of mathematics in the early childhood Montessori classroom begins with indirect preparation and can then be divided into eight areas of work: (a) numbers to ten, (b) introduction to the decimal system, (c) teens and tens, (d) simple counting, (e) memory work, (f) fractions, (g) decimal system operations, and (h) activities leading to abstraction.

Indirect Preparation

The exercises of practical life and the sensorial materials indirectly prepare the child for mathematical thinking in the Montessori preschool classroom. Practical life activities involve the children in caring for themselves and the environment and provide indirect preparation for mathematics by developing the child’s concentration, coordination, sense of order, and logical and sequential thought patterns. The sensorial materials allow children to classify sensorial impressions in an organized, orderly manner and enable the child to work in quantities from one to ten in several dimensions. Activities in both areas help the child to move with precision and to work toward exactness of movement and thought.

Numbers to Ten

The activities in the first area of work, numbers to ten, introduce the child to units of quantity using a sequence of materials that increases in difficulty and slowly leads the child to a conceptual understanding of number. A set of ten wooden rods in graded lengths from one decimeter to one meter is used to help the child learn the names of the numbers and that each number represents a quantity separate and distinct from all the others. The rods are painted red and blue in alternate decimeters - the number of partitions in each rod represents the number of the rod--and the child is given the name for two or three numbers at a time through activities involving manipulation of the rods. A set of sandpaper numerals is then used to introduce the symbol for the quantities the child has come to recognize and eventually a presentation is given to associate the quantities represented by the number rods with their written symbol.

A material called the spindle boxes is then demonstrated to the child to illustrate number as a collection of items and to introduce zero. By placing the corresponding quantity of spindles in compartments marked from zero to nine the child is provided with further experience in counting and in associating quantity and numeral. The fixed order of the numerals helps the child to learn the numbers in sequence and prepares her for the next exercise - the counters and numerals - in which both the order and the quantity must be established by the child. A bead stair containing bead bars of different lengths and colors ranging from one to nine can also be introduced to provide further sensorial experience with counting units.

Once the child is able to count independently and has some recognition of number outside of a sequence, she can be asked to retrieve a specific number of objects from a different location in an activity called the memory game. This final exercise in the numbers to ten sequence provides the child with practice in counting and helps her to develop a memory of the numbers from zero to ten.

Although many Montessori teachers supplement these basic exercises with additional zero to ten counting activities in their programs, the materials described here form the basic core for learning the numbers of our base ten system and help the child to construct an understanding of number through direct sensorial experience with concrete materials.
Decimal System Introduction

Two sets of materials, the golden beads and the number cards, introduce the child to the quantities and symbols of our decimal system. The golden bead material consists of individual beads to represent units and beads strung together on wire to make up 10 bars, 100 squares, and 1,000 cubes. The number cards consist of four sets of cards representing the hierarchy of numbers in the base ten system and are as follows: (a) one to nine in green, (b) 10 to 90 by tens in blue, (c) 100 to 900 by hundreds in red, and (d) 1,000 to 9,000 by thousands in green.

The child is first introduced to the decimal quantities of 1, 10, 100, and 1,000 through a naming activity using a unit bead, a ten bar, a hundred square, and a thousand cube. The written symbol for each quantity is then presented to the child using the unit, ten, hundred, and thousand number card and finally, a demonstration is given to associate the quantities with their symbol. Once the child is familiar with the decimal categories, a lesson is given to illustrate that in order to go beyond nine in any one category, it is necessary to go to the next higher category.

Further sensorial experience with the decimal system and its numerals is provided through a series of layouts. During the first layout, the quantity represented by the number cards is laid out in a grid formation using a large quantity of the golden bead material. The number cards are laid out in a similar fashion in the second layout, and in the third, both quantity and symbol are combined to introduce the association between quantity in the decimal system and its symbol. These exercises provide the child with a visual representation of the decimal system and its relative proportions and reinforce that if you go beyond nine in one category you need to go on to the next category. Once the child has experienced the full decimal layout, she can begin to compose numbers by combining number cards and finding the corresponding quantity or by placing quantities from more than one category together and locating the appropriate number cards.

Teens and Tens

Activities in the next area of work, teens and tens, run parallel with the early decimal system presentations and aid the child in her counting and construction of numbers between 11 and 100. During the first activity the child constructs both the quantity and symbol for the numerals from 11 to 19 using a material called the teen boards. After the child has practiced with the teen boards, the ten boards are presented to introduce the concept that units can be added to tens and to explore the numbers from 11 to 99.

Practice in counting from one to a hundred is provided through a material called the hundred board and through a set of activities called linear and skip counting. The hundred board enables the child to place tiles with the numerals from one to one hundred printed on them on a grid board according to a control chart that illustrates the correct layout of the tiles. Linear counting involves the child in counting and labeling different bead chains from the bead cabinet material which consists of short chains of bead bars representing each numeral squared and long chains of bead bars representing each numeral cubed. Skip counting is undertaken by reading the labels that have been placed by the child while linear counting. For example, once the five short chain has been counted and labeled, the labels, placed next to the last bead in each bar, can be read back - 5, 10, 15, 20, and 25 - to demonstrate skip counting. These activities provide the child with practice in counting and indirectly prepare her for later activities in multiplication, squaring, and cubing.
**Simple Counting**

A variety of activities that prepare the child for addition, subtraction, multiplication, and division make up the section of work entitled simple counting. Although the child will have already experienced addition sensorially by combining various materials such as the number rods, the child’s first formal introduction to the concept of addition is given through two materials - the snake game and the addition strip board. During the snake game, the child places colored bead bars of different lengths in a zig-zag formation and turns the snake golden by counting the beads up to ten and replacing each group of ten with a gold ten bar. Any remaining beads in a bar are counted and a place holder procedure is introduced. By replacing the colored bead bars with the gold ten bars the child visually experiences equivalence and is familiarized with the possible number combinations equal to ten.

The addition strip board consists of a hand-board chart divided into squares with the numbers 1 to 18 printed across the top. Two sets of numbered strips, one in red, the other in blue, are arranged on the board in different formations to help the child learn the sequence of addition combinations from one to nine. The subtraction strip board is used in a similar fashion and provides practice with the subtraction tables from 1 to 18. However, the subtraction strip board is a more advanced material and may not be introduced until the elementary years.

Multiplication and division are introduced to the preschool Montessori child using the multiplication board and the unit division board. By distributing small beads on each of the perforated boards, the child kinesthetically experiences multiplication and division and is helped to develop a memory of basic multiplication and division combinations. Further practice with multiplication is provided through a series of bead bar layouts that sensorially and visually demonstrate the multiplication tables and equivalencies.

**Memory Work**

The child is assisted in her memorization of arithmetic facts through a collection of charts called the memorization charts. For each operation there is a set of charts that helps the child progress from a practicing stage to complete memorization of combinations. As the child becomes more proficient in each operation, the chart they work with becomes more abstract until the final chart, which is blank, is presented and the child must fill in the answers herself from memory.

**Fractions**

A sensorial introduction to the concept of fractions is given to the youngest children in the Montessori preschool classroom through four large wooden skittles that represent a whole, halves, thirds, and fourths. The tactile and visual exploration of fractions is then continued through the fraction circles that consist of ten metal frames each containing a 10 cm circular inset. One inset is a whole circle while the other circles are divided respectively into two, three, four, five, six, seven, eight, nine, and ten equal parts. Each circle segment has a knob enabling the child to remove and replace the pieces. The circle pieces can then be used to introduce the names of the fraction parts and can be employed in a variety of activities to prepare the child for the advanced fraction exercises she will encounter during the elementary years.
Decimal System Operations
The child’s work with the decimal system continues through a series of activities exploring the processes of addition, subtraction, multiplication, and division. The same material, including a bank of golden bead material and three sets of number cards, is used to carry out all the exercises and a similar format and progression is followed for each operation.

Static addition, which does not require carrying from one category to the next, is presented first and should be given to a group of children who are experienced with associating large quantities with their numerical symbols. A problem is presented, e.g. 3,326 + 2,431, and the children are asked to obtain the number cards and corresponding quantity of material for each addend. The quantities are then combined and the answer is laid out using the third set of number cards. In dynamic addition, the next presentation, the procedure is the same except the problem given will require regrouping from one hierarchy to the next. Both exercises give a sensorial impression of addition as a putting together of quantities, and reinforce the concept of place value.

Static and dynamic subtraction, multiplication, and one digit division follow the same process while two digit division is presented using only the bead material in a related manner to provide the child with a sensorial introduction to long division.

Activities Leading to Abstraction
The final group of exercises in the early childhood Montessori math curriculum provide further experience with arithmetic and move the child closer to abstraction and symbolic representation. Three activities - the stamp game, the dot board, and the small bead frame - make up the area of work that leads the child to abstraction. While each of these materials is initially introduced using addition, they can also be used to perform subtraction and multiplication, and in the case of the stamp game, division.

The stamp game is used by an individual child to do addition in the same manner as in the decimal system operations only “stamps” are used to represent the golden bead material in the following manner: (a) green stamps with 1 written on them represent units, (b) blue stamps with 10 printed on them stand for a ten bar, (c) red stamps with 100 written on them represent a hundred square, and (d) green stamps with 1,000 written on them stand for a thousand cube. The dot game is also representational and introduces the child to column addition and the decimal category of 10,000 using paper specifically prepared for this exercise.

Rather than combining the quantities of two addends using the golden bead material or stamps, in this activity the child combines two amounts by placing different colored dots representing the decimal categories in the appropriate columns. Dynamic addition is carried out by crossing out any rows of ten found in a column and placing a one for each row in the next higher category.

A further step toward abstraction of addition is taken through the use of the small bead frame that consists of a wooden frame with four wires across, each strung with 10 beads and representing the hierarchy of numbers from units to thousands. A problem is laid out by sliding the corresponding beads from the left of the frame to the right for one addend beginning with the units and then adding to these each category from the second addend and carrying while doing so if necessary. The child may then count the beads in each category on the right of the frame to arrive at the sum. The procedure is less time consuming than the previous addition exercises and directly prepares the child for abstract computation.
The Elementary Mathematics Curriculum

Montessori mathematics at the elementary level begins with the Story of Numbers and can then be divided into 13 primary areas of work: (a) numeration, (b) multiplication, (c) division, (d) fractions, (e) decimal fractions, (f) squaring and cubing, (g) square root and cube root, (h) powers of numbers, (i) negative numbers, (j) non-decimal bases, (k) word problems, (l) ratio and proportion, and (m) algebra. The beginning work in most sections is introduced during the child’s first few years in the elementary classroom while some of the later activities may not be presented until the fifth or sixth grade level. General age levels for lessons are occasionally given and prerequisite work is described where appropriate.

The Story of Numbers

Mathematics at the elementary level is introduced and explored from a historical perspective through a narrative called The Story of Numbers. The Story of Numbers presents an overview of early number systems from the Mayans to the Romans and investigates the origins of our present system of numeration. The history of mathematics provides a foundation for all the child’s work in mathematics and serves as a means for relating math to every other area in the Montessori curriculum. Students are encouraged to initiate their own follow-up after the story has been told.

Numeration

The area of work called numeration explores our decimal system and its properties beyond the child’s experiences in the preschool Montessori classroom and sensorially introduces multiples, factors, and the concept of measurement.

An expanded view of our decimal system is presented both physically and visually through the wooden hierarchy material that consists of seven geometric forms representing the hierarchy of numbers from a 1/2 cm unit cube to a 50 cm million cube. This dramatic material emphasizes the relative size and shape of one category to another and is color-coded (green cubes, blue bars, and red squares) to introduce families of numbers, i.e. the simple family, the thousand’s family, and the million’s family. A set of numeral cards with 1, 10, 100, 1000, 10000, 100000, and 1000000, printed on them is included to familiarize the child with the symbols for the categories represented by the material. This material gives a clear visual representation of the hierarchy of numbers and directly prepares the child for working with numbers beyond the thousands category.

The commutative and distributive laws of multiplication are brought into the child’s consciousness through a series of exercises using the colored bead bars from 1 to 10, number cards from 1 to 9, and later the golden bead material for multiplying with numbers greater than units. Multiples and factors, including activities introducing the lowest common multiple and the highest common factor, are explored using colored pegs and a peg board, while measurement is examined through a story of its history and a variety of exercises focused on length, volume, weight, time, angles, money, temperature, etc. Work in all of these areas provides sensorial experience with the properties of our number system and either directly or indirectly prepares the child for the later mathematical work they will encounter in the curriculum.
Multiplication

Multiplication at the elementary level builds on the child’s experiences with the decimal system operations and the small bead frame in the preschool Montessori classroom and is investigated primarily through the use of four materials: (a) the large bead frame, (b) the checker board of multiplication, (c) the flat bead frame, and (d) the bank game.

The large bead frame is used in the same manner as the small bead frame but consists of seven strings of ten beads representing the hierarchy of numbers from units to millions rather than four strings representing units to thousands. The expanded size of the frame allows the student to practice reading and writing large numbers and helps to reinforce place value and that 10 in one category makes up one of the next higher category. A series of activities is completed on the frame culminating in an exercise introducing two-digit multiplication both sensorially and as written out through specifically prepared notation paper.

The checker board of multiplication provides additional manipulative experience with long multiplication and moves the student toward abstraction by becoming less concrete as the child gains proficiency with her multiplication facts. The checker board exercises lay a foundation for completing a geometrical form of multiplication on graph paper and indirectly prepare the child for the squaring of numbers, square root, and algebra.

The flat bead frame, in conjunction with the bank game, is introduced as the last step in leading the child toward abstract multiplication. Whereas the large bead frame contains seven strings of colored beads, the flat bead frame has nine wires of ten golden beads and is manipulated to provide the child with the partial products of a problem as they are recorded on paper. As the student becomes more able to calculate in her head, use of the frame becomes unnecessary and a step closer to abstraction of multiplication will have occurred. The bank game, used by a group of children, provides additional practice with long multiplication using a lay out of number cards from 1 to 9 million, and reinforces the changing process and what is meant by category multiplication. Both materials help the child to construct her own knowledge of the multiplication process and are used over an extended period of time to meet the individual needs of each student.

Division

Both distributive and group division is introduced at the elementary level through a series of exercises that bring the child to an abstract understanding of division. Distributive division, in which a quantity is shared so that each unit receives the same amount, is explored through a material called the racks and tubes. By distributing different colored beads representing the categories of numbers on one or more division boards while using the racks and tubes the child is able to perform short and long division and is introduced to the recording process that will eventually enable her to divide abstractly. The student’s work with distributive division is followed by an introduction to group division in which the answer is derived by subtracting the divisor from the quantity as many times as it takes to be used up. The stamp game material previously used in the primary classroom is used to concretely illustrate this process and to help the child with the estimation procedure necessary for completing long division on paper. A series of divisibility exercises is also explored in this area of work to acquaint children with the rules of divisibility and to excite interest in the study of mathematics.
Fractions
The fraction metal insets introduced at the primary level are used as a starting point for almost every fraction concept explored in the elementary classroom. Beginning with an introduction to the quantity, symbol, and language of fractions, and progressing to activities of equivalence, simple operations, adding and subtracting with different denominators, and various multiplication, division, and word problem exercises, the metal insets provide a sensorial basis and sense of continuity for all the fraction work undertaken by the child at this level. Once a concept has been introduced using the insets, the student may continue to explore using the material or choose to create her own booklet or chart as a follow-up activity. The sequence of presentations helps to ensure that the child is adequately prepared for an activity before it is introduced and slowly moves the child toward more difficult exercises as an understanding of the more basic concepts is obtained. A set of charts illustrating different fraction concepts can be hung on the wall during the child's study and used for recalling information as necessary.

Decimal Fractions
Decimal fractions are introduced after the student has had experience with multiplying and dividing and has worked with the fraction insets. Small wooden cubes color coded to represent the hierarchy of decimal numbers from one-tenth to one-hundred thousandth are used throughout the study of decimal fractions in combination with the decimal fraction board—a working board containing columns representing whole and decimal numbers on which the decimal cubes, corresponding numeral cards, and colored whole number beads may be placed. Once a set of initial presentations introducing the quantity and symbols for decimal fractions has been given, the decimal board is used to introduce the formation and reading of decimal numbers and to perform addition, subtraction, multiplication, and division operations. The hierarchy of numbers as existing even in decimal fractions—that ten of this category equals one of the next higher—is emphasized through these operations, and the student is shown how to complete the processes on paper. Further experience with multiplying whole and decimal numbers is provided through the decimal checker board. A procedure for determining the rule for multiplying and dividing decimal fractions can then be demonstrated as well as an activity for converting common fractions to decimal fractions.

Squaring and Cubing
The child’s work with squaring and cubing in the elementary classroom begins with a sequence of activities introducing the squares and cubes of numbers using the bead cabinet material which consists of different colored bead chains, squares, and cubes for each number, 1 through 10. Once the concept, notation, and numerical values first for squares, then for cubes, has been presented with the bead material, the child can be shown a series of games and exercises that explore squares, cubes, and the multiplication tables both sensorially and numerically. Different power scales and decanomial layouts can be shown to the child as well as methods for adding, subtracting, multiplying and dividing the squares and cubes of numbers. These activities are used to arouse the child’s interest and to provide a basis for the more structured squaring and cubing exercises and the student’s later work in algebra.

An extensive series of activities exploring binomials and trinomials both numerically and algebraically continues the child’s work with squaring and cubing. An assortment of materials, including the bead squares, the
colored bead bars, the golden bead material, and the colored pegs and peg board is used to progress through the squaring exercises while the wooden cubing material, which consists of one cube and 27 squares for each of the powers from 1 to 9, is used for many of the cubing activities. The binomial and trinomial cubes previously introduced as sensorial puzzles in the preschool Montessori classroom are reintroduced at this level to demonstrate cubing algebraically and a second trinomial cube, color-coded to represent each category value from a unit to ten thousand, is used to illustrate cubing a trinomial hierarchically. These exercises allow the student to discover relationships between the component parts of squares and cubes and prepare them for abstract analysis of polynomials.

**Square Root and Cube Root**

A conceptual understanding of square and cube root is attained by the student in the Montessori classroom through a succession of activities using the colored pegs and peg board to investigate square root and the wooden cubing material to examine cube root. A lesson introducing the concept, language, and notation begins both the study of square root and cube root and in both areas of work the child moves from concrete exploration with the materials toward discovering the algorithm necessary for completing the process on paper. The manipulatives enable the student to find the square and cube roots for progressively more difficult problems and by paralleling the algorithm in their use, prepare the child for arriving at her own understanding of the abstract procedure. The exercises also provide the child with practice in place value and multiplication.

**Powers of Numbers**

The student's work with the powers of numbers begins after the exercises with squares and cubes through the use of a material called the power of two cube. Consisting of cubes and prisms, each a progression of squaring the previous piece, the cube concretely demonstrates the powers of two and is used to introduce the terminology of base and exponent. Bases other than two are examined using a large quantity of small white cubes, and the hierarchical material is reintroduced as demonstrating the powers of ten. The child can then be shown how to complete operations using exponential notation and can be lead to discover the rule for multiplying and dividing numbers of the same base.

**Negative Numbers**

Negative numbers are introduced in the elementary classroom through a variation on the snake game that is presented at the primary level to reinforce the process of addition. Negative bead bars are included in the procedure during the negative snake game to visually demonstrate that negative numbers decrease the quantity during addition and to emphasize that numerically equal positives and negatives cancel one another out. Once the child has sensorially experienced negative numbers, she is shown how to record the snake on paper and is later introduced to subtracting, multiplying, and dividing negative numbers using concrete materials. The student continues to work with the manipulatives until they are no longer needed and is encouraged to derive the rule for completing operations with sign numbers through her own explorations.
Non-decimal Bases
Non-decimal bases are introduced after the child has worked extensively with the decimal system and is aware of the geometric shapes of the various powers and that we have a place value system where zero is necessary. The numeration of non-decimal bases is examined by counting with the bead material on a number base board that has been divided into four categories: (a) units, (b) bars, (c) squares, and (d) cubes. For example, when counting in base five, unit beads up to four are placed in the unit column and recorded one at a time as 1, 2, 3, 4. As a fifth unit bead is added, the five units are exchanged for a five bead bar, which is placed in the bar column, and the number 10 is recorded to illustrate one bar and no units. As another unit is added to the unit column the number 11 is recorded and so on. Once the student is comfortable with counting in non-decimal bases, a series of charts can be presented to assist the child in performing operations in bases other than our own, and techniques for converting from a given base to base ten and vice versa can be presented. These exercises are presented to arouse the interest of the child and to expand their perception of what is meant by a number system.

Word Problems
Word problems should ideally be introduced through practical applications in the classroom but may be developed if they don’t occur on their own. The student should be given the steps necessary for solving a problem and once carried out, the answer should be checked against what is known to determine if the solution makes sense.
Two kinds of problems-velocity, distance, and time; and interest, rate, time, and principal-are included in the elementary curriculum. These are given to the child in three levels beginning with a sensorial introduction at an early age where labeled tickets and the golden bead material are used to help the child set up the problem and determine the solution for each variable included. At the second level the student is helped to identify more precisely what was done arithmetically, and at the third level, the rule or formula for the problem is presented.

Ratio and Proportion
The child’s study of ratio moves from a concrete introduction using the colored pegs on the peg board though a series of exercises that explore the arithmetical recording of ratio, ratio written as fractions, ratio stated algebraically, and various word problems where practical applications of ratio are investigated. Proportion is introduced after the student has studied ratio and is able to balance equations. Various objects in the environment such as a one bead bar and a five bead bar, geometric figures which are equal in proportion, the power of two cube, etc., are used to illustrate proportional relationships and to provide a foundation for calculating arithmetically and algebraically with proportion. The student’s work in this area is completed by applying proportion to word problems or in problem situations occurring naturally in the classroom.

Algebra
Children are introduced to algebra in the Montessori classroom once they are able to write formulas in word problems and can perform operations in fractions and negative numbers. In the first series of activities the child is shown how to balance an equation that has been laid out in bead bars when something has been
added or subtracted to one side or when one side has been multiplied or divided by a number. The student then explores operations with equations and is introduced to algebraic word problems. These activities provide concrete experience with statements of equality and form a basis for the child’s later work with algebra.
Great Story:

Story of Our Numerals

**Purpose:** To give children the notion that the idea of number is present in human society.
To introduce the child to the history connected with our numerals.

**Materials:** The story, clay and stick, charts.

**Notes:** This is a “Great Story” presented to all children new to the Montessori elementary class. It follows the great story of our alphabet, but not by too many days. There is no need to delay early presentations in mathematics to children new to the classroom until the story has been presented. However, do not delay the story beyond the first six weeks of class.

The story of our numerals should be given as a whole into which details will later be fit. The story should appeal to the imagination.

**Sequence of Important Points:**

1. Recall the previous three great stories, plus the “Story of Our Alphabet”.
2. More than likely, people spoke words that meant numbers. They also probably used some form of tallying, such as with sticks and stones.
3. Present examples of recording numbers: Sumerians and Egyptians.
5. Discuss how the invention of printing led to number symbols as we have them today.
6. Leave the child to ponder.

**Presentation:**

Remember we talked about how the universe came to be with all its particles following their laws. Then life came with its laws. Then human beings with their gifts of mind, love, and hands lived on earth. Some time ago I told you the story of the Phoenicians, who invented the symbols that eventually became our alphabet. Today I’m going to tell you the story of our numerals, the way we write numbers. There were many people involved.

While it is uncertain how long ago humans began to use speech, it is conceivable that when they did, they used words that expressed quantity. Whether to count how many people in a cave, how far it was to the river, or to take a particular measurement, there was the same need then to communicate using numbers as there is today.

People who have studied languages have found that all have some idea of number even if only the words one and two could be found in the vocabulary. In one tribe in Bolivia, no specific words for numbers existed
except the word “alone” used to represent one. In languages where only a few numbers were used, there was little or no need to express large quantities.

As there were no written records when speech developed, it is impossible to know how the use of number began. One early need for numbers was in counting. The variety of things used to count with was endless ranging from sticks, pebbles, shells, fruit, and knots on a string, to the universal use of the upright finger. One tribe, the Malayas, used stones to show quantity when the amount exceeded that which could be shown on the fingers.

**Sumerians and Babylonians**

People spoke for many years before words were put into writing. In the same way, it took many years before signs for numbers were developed. The first records for written numbers were made about 5,000 years ago in the Asian valley of Mesopotamia located between the Tigris and Euphrates Rivers. About 2,000 years later, the Sumerians, living in the same area, developed a system of writing numbers known as cuneiform. Its use spread and was adapted by the Babylonian merchants who used it to keep records of their trade. Using a stick with its point shaped into a triangle, the Babylonians made impressions in slabs of clay that were baked until they were hard.

**CHART (Babylonian Numerals)**
Egyptians

The ancient Egyptians living near the Nile River in Africa were also merchants and tradesmen who needed to keep records of their transactions. As they became more prosperous, the need for writing large numbers prompted the development of a system extending into the millions. Using what is referred to as picture writing, the Egyptians picked things in their environment to symbolize the base ten categories of numbers. While in our system numbers are read from left to right, the Egyptians alternated from left to right on one line, then from right to left on the next in the same way they would plow their fields.

CHART (Egyptian Numerals)

Chinese

The oldest known numerals were first used by the Chinese and were later adapted by the Japanese. The system contained symbols for the numbers 1 to 9 and for the 10s and 100s and 1000s categories. Rather than writing their symbols horizontally, the Chinese wrote vertically reading from top to bottom. The first symbol in a number was used to designate the quantity of the second symbol, with the third designating the amount of the fourth and so on.

CHART (Chinese Numerals)
Greeks

The early Greeks developed a system using the first letters of the number’s names for their symbols. The 5 equivalent was their penta letter, the 10 used the deca letter, the 100 hecto, and the 1,000, kilo.

**CHART (Greek Numerals)**

Romans

The Romans used a similar system still in use today in the copyrights of books. Some symbols stood for the first letter of the number words, such as the “C” which came from the word centum meaning 100, and the “M” from the word mille standing for 1,000. Others may have developed from hand signals. For example, the “V” for 5 may have represented the single hand held with the thumb and forefinger apart, and the “X” for 10 may have symbolized two hands held together with thumbs crossing in an x. The “D” for 500, may have evolved from half of the shape used for 1000 before the “M” was used. The Roman method was used throughout Europe for bookkeeping until the 18th century partially because of the simplicity it afforded for addition and subtraction problems.

**CHART (Roman Numerals)**
Hindus

The origins to our present system can be traced 1200 years ago to the Hindus. On their travels to India for trade, the Arabs encountered an arithmetic book written by the Hindus and translated the system for their own use. The book eventually turned up in Europe and was translated in Latin. As it was hand written in manuscript form and more difficult to write than Roman, the system didn’t get passed around quickly, and as it did, it varied considerably with different handwritings. Finally, in 1415, the printing press was invented making it less easy to change the printed symbols. With more scientific work occurring, there came a need for calculating rather than just bookkeeping. A second look at the Hindu system with its zero was taken, and it was adapted for use particularly with problems where place value was important.

Since the book that brought these numerals to Europe came from Arabia, they were called the Arabic numerals.

One symbol which has been constant throughout time and is like ours today, is the symbol used for the number ”1”. Whether horizontal or vertical, all ancient peoples as well as the Greeks and Romans had a symbol similar to ours. The symbols for 2 and 3 have also resembled ours and may have resulted from a hurried attempt to make 2 or 3 slashes on their sides thus resulting in their connectedness.

CHART (Changes in Our Numerals)
Reference
Evolution of Numerals

Brahmi

Indian (Gwallor)

Sanskrit-Devanagari (Indian)

West Arabic (Gobar)

East Arabic

11. Cent. (Apices)

15. Cent.

16. Cent. (Dürer)

Cod. Vigiliius (976 C.E.)

Vatican MS. lat. 3101 (1077)

British Mus. Add. 17808 (XII)

General forms, c. XIII

General forms, c. XIV

General forms, c. XV

General forms, c. XVI
Greek *acrophonic* numbers: (used ca. 700/600-100 B.C.)

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Greek *alphabetic* numbers:

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Traditional Chinese Counting Rods

Fig. 24.61. Origin and evolution of the number 3. (For Arabic and European numerals, see Chapters 25 and 30.)
Fill in the Roman Numerals

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## II Numeration

### Contents

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Exercise One: Number x a Number (Commutative Law)

Exercise Two: A Sum Multiplied by a Number \((1 + 1) \times 1\)

Exercise Three: Multiplication of a Sum by a Sum \((1+1)\times(1+1)\)

Exercise Four: Multiplication of a Sum by a Sum with Signs & Cards

  - Part A – Introduction of Operational Signs
  - Part B – Introduction of Cards
  - Part C – Using Cards Only
  - Part D – Completing on Paper

Exercise Five: Multiplying with Numbers Greater than Units

  - Part A
  - Part B
  - Part C
  - Part D

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Introduction to Measurement

History of Measurement

Measurement of Length

Exercise One: Introduction to Metric System

Exercise Two: Introduction to Units of the Metric System

Presentation:

Exercise 3: Exchanging Other Measurements
Wooden Hierarchy Material

**Introduction:** This lesson opens the door to the rest of the number system through further exploration and investigation of the decimal system. The material gives a physical and visual impression of numbers while summarizing the child’s previous experience in the primary class.

**Material:**
- Wooden cubes, rods, squares and blocks, colored as follows:
  - 3 Green = units, thousands, millions, 1/2 cm, 5 cm, 50cm
  - 2 Blue = tens, ten thousands, ten millions, 1/2 cm x 1/2 cm x 5 cm and 5 cm x 5 cm x 50 cm
  - 2 Red = hundreds, hundred thousands, hundred millions, 5 cm x 5 cm x 1/2 cm and 50 cm x 50 cm x 5 cm
- Each quantity starting with the second is 10 times the proceeding one. The four smallest pieces are kept on a tray.
- A set of white cards approximately 3 cm x 9 cm with written numerals 1 to 1,000,000 (1, 10, 100, etc.).

**Presentation 1: First Arrangement**

1. Gather children around the rug with the wooden materials from 1 to 1,000 displayed, keeping the higher numbers out of sight until you need them.
2. Hold up the green unit cube and state, “This is one unit.”
3. Place to the far right of the mat.
4. Hold up the blue ten bar and state, “This is one 10. There are 10 units in one ten bar. 1, 2, 3, …” Verify by counting; place vertically to the left of the unit cube.
5. Hold up the red hundred square and state, “This is 100. How many 10 bars are there in 100? Count to verify. Place to the left.
6. Hold up the green thousand cube and state, “This is 1,000. There are 10 hundreds in 1000. 1, 2,…” Verify by counting. Place to the left.
7. Ask, “What would happen if we put 10 thousands together?” Bring out the blue ten thousand rod and state, “This is 10,000. This is 10 times larger than 1,000. It represents 10 thousand cubes.” Count to verify.
8. Ask the children what they think the next object will look like. What number will it represent? What color will it be? What shape will it be? How big will it be?
9. Bring out the red 100,000 square and discuss the accuracy of the group’s predictions. Repeat the above for the green million cube.
10. Ask: “Do you notice anything about the color?...Each color has a different shape – a cube, bar or square.” Introduce the first group as the simple family, the second as the thousands family, the third as the millions.
11. State, “So we have a unit, a ten, and 100…(etc.) Show me 1,000. Where is 100,000?…” Continue with a 3 period lesson.
Presentation 2: Second Arrangement
1. Hold the unit cube and place it at the bottom lower right corner of the rug. Name the unit cube.
2. Hold the 10 bar and place it to the left of the unit cube. Name the 10 bar.
3. Hold the 100 square and place it to the left of the ten bar. Name the 100 square.
4. Hold the 1,000 cube and place it above the unit cube. Name the 1,000 cube.
5. Hold the 10,000 bar and place it above the 10 bar. Name the 10,000 bar.
6. Hold the 100,000 square and place it above the 100 square. Name the 100,000 square.
7. Hold the 1,000,000 cube and place it above the 1,000 cube. Name the 1,000,000 cube.
8. Talk about the families and the patterns: cube, line and square.

Presentation 3: Introduction to Symbols
1. Place the numerical value or symbol for each wooden material and introduce, e.g. “This is one unit. This is one ten. How many zeros does one 10 have? Here we have 100 – how many zeros are there in 100?…”
2. Discuss using a comma for numerals greater than 100: “If we were writing these numbers we would use a comma to show the thousands family. What’s before the comma?…”
3. Mix the symbols and have the children place them where they belong.

Purpose:
1. To give a clear visual relationship of the hierarchy in these categories as well as being an indirect preparation to geometry. This is a good preparation for the long bead frame.
2. A sensorial presentation for the quantity greater than 1000.
3. Reinforcing the color scheme.
4. Gives a geometrical progression from a point, line cube and plane.

Age: 6 years old
Commutative and Distributive Laws of Multiplication

**Material:**
- Box of assorted colored bead bars; white and gray cards from 1 to 9 the same size as decimal cards; several slips marked +, −, x and =.
- Set of small decimal cards 1 to 3,000 (colored cards); 6 or 8 coin envelops, brackets, little slips of paper (2 cm sq.); felt mats to work on.

**Introduction:** These activities use knowledge of multiplication in a game like fashion conveying the idea that it can be fun to do. This series brings to the child’s consciousness some properties of the number system – in particular

**Commutative Law of Addition**

\[ a + b = b + a \]

Addition is commutative. This means that the order of adding any two numbers does not affect the result.

**Associative Law of Addition**

\[ (a + b) + c = a + (b + c) \]

Addition is also associative. This means that the order of grouping numbers together does not affect the result in addition.

**Commutative Law of Multiplication**

\[ a \times b = b \times a \]

The commutative law of multiplication states that the order of the multiplicands does not affect the answer, just as is true in addition:

\[ 3 \times 5 = 15 \text{ or } 5 \times 3 = 15 \]

The numbers can change places just as you can when you commute to school on the bus.

**Distributive Law of Multiplication over Addition**

\[ a \times (b + c) = ab + ac \]
Exercise One: Number x a Number (Commutative Law)

1. Hold up a 5 bar. Place on the mat horizontally.
2. State, “I’m going to do a multiplication. I’m going to take this 5 bar 3 times.” Place a gray 3 card on the mat to the right of the 5 bar.
3. “5 taken one time, two times, three times. What is 5 taken 3 times worth?” Place 3 5 bars under the 5 bar and the answer below vertically using a 10 bar and a 5 bar.
4. “Now let’s take 3 5 times. 3 taken once, twice…what is 3 taken 5 times worth?”
5. Place a 3 bar and gray 5 card to the right of the 5 bar and 3 card; place 5 3 bars below the 3 bar and the answer, 15, in beads below the 5 card.
6. Note that 5 x 3 = 15 and 3 x 5 = 15. State: “I wonder if it would work with 2 other numbers. Would you like to try?”
7. Later give the definition of the commutative law of multiplication.

Exercise Two: A Sum Multiplied by a Number (1 + 1) x 1

1. State, “We’re going to do more multiplication. Look what’s inside this envelope. We’re going to take this 3 times.”
2. Place a 2 bar and 5 bar horizontally in brackets; place a 3 gray card to the right.
3. “Because the 2 and 5 are together, we’ll put brackets around them. We’ll multiply this by 3.”
4. Place 3 2 bars below the 2 bar then 3 5s below the 5 bar. State, “2 taken one time, 2 times, 3 times. And now 5 taken one times, two times, 3 times. How much is this all together?”
5. Place the answer (21) vertically below in bead bars.
6. “Now we’ll multiply what’s in this envelope by 3.” Lay out a 3 bar to the right of the first lay out. Place a 2 and 5 gray card in brackets to its right.
7. “What’s our answer? Look, we got the same answer! I wonder if this will work with other numbers. Would you like to try?”
8. Place answer below second lay out.

Purpose: To bring to the child’s attention the concept of multiple terms in the multiplicand or multiplier, and the use of brackets.
Exercise 1

Exercise 2
Exercise Three: Multiplication of a Sum by a Sum (1+1)x(1+1)

1. State, “This time we’ll multiply what’s in this envelop by what’s in this envelop. We’ll set this up like this”
2. “We’ll turn this card over(5) and begin by multiplying by 3. 2 taken 3 times is ..... 5 taken 3 times .....” Place 3 2 bars below the 2 bar and 3 5 bars below the 5.
3. “Now we’ll multiply by 5 (turn both cards over). 2 taken 5 times .... 5 taken 5 times....” Place 5 2 bars below the 3 card and 5 5 bars below the 5 card.
4. “Let’s find our answer. Here we have 6 (place a 6 bar below the 3 2 bars). What do we have here? (15) an here? (10) here? (25)” Place answers below each quantity.
5. “We can add them all together for our final answer. (56)
6. Place in bead bars of to the right of the sums.

Purpose: For a further awareness of the distributive principle; stress you must multiply each term of the multiplier for each term of the multiplicand. This is indirect preparation for binomial multiplication.

Exercise Four: Multiplication of a Sum by a Sum with Signs & Cards

Part A – Introduction of Operational Signs

Repeat multiplying a sum by a sum (4 and 2 bead bars in one envelop, 3 and 5 gray card in another) adding operational signs where appropriate by talking through what you’ve been saying and how signs will express this. Read as (4 + 2) x (3 + 5) =

Work the problem as usual; use decimal cards for answers combining and exchanging for the final answer.

Part B – Introduction of Cards

Lay out as above; state that you will write what you’ll do before using the beads. Lay the problem out in expanded form using white number cards. Place beneath the problem in order:

Part C – Using Cards Only

Work out completely without beads using cards; the child can be shown horizontal addition.

Part D – Completing on Paper

The child works the problem out on paper. This can be checked by adding the numerals in each bracket and multiplying them.

Purpose: Preparation for algebra.
Excercise 3

Excercise 4
Exercise Five: Multiplying with Numbers Greater than Units

The child must have completed exercises 1 to 4 and understand the process.

Part A

– Using the Golden Bead Material

1. On a longish slip of paper write the problem: 42 \times 23. Place in the middle at the top of the felt mat.
2. Lay out in cards, symbols and brackets:

\[
\left( \begin{array}{c}
40 \\
+ \\
2
\end{array} \right) \times \left( \begin{array}{c}
20 \\
+ \\
3
\end{array} \right)
\]

3. Introduce as the multiplier and multiplicand.
4. Turn the 3 card over. State that you'll multiply 40, 20 times. Lay out 4 ten bars horizontally:

\[
\left( \begin{array}{c}
40 \\
+ \\
2
\end{array} \right) \times \left( \begin{array}{c}
20 \\
+ \\
3
\end{array} \right)
\]

5. Lay out 19 other groups of 4 ten bars underneath to make 4 groups of 20 leaving a space between groups of ten.
6. To the right of this work out 2 taken 20 times using unit beads.
7. Continue with 3 \times 40 under the groups of ten bars and 2 \times 3 under the units.

\[
\left( \begin{array}{c}
40 \\
+ \\
2
\end{array} \right) \times \left( \begin{array}{c}
0 \\
+ \\
3
\end{array} \right)
\]

8. Talk through and exchange groups of 10 10s for hundreds, and the lines of unit 10s for 10 bars.
9. Discuss what categories you end up with when multiplying different categories together, i.e. 10s times 10s gives you 1000s, etc.
10. Find out the answer by counting and placing out decimal cards for each section; add partial products, exchange as needed and place the answer after the equal sign.
11. The children can repeat with their own problems.

Part B

– Using Cards and Beads

Same procedure; expand the problem in cards but still lay out the beads.

Part C

– Completing on Paper

Complete the problem on paper.

Part D

– On Paper with 3 Digit Numerals

Introduce 3 digit numerals on paper helping as needed.
Exercise 5

\[ 40 \times (20 + 3) \]

\[ 40 \times 3 \]

\[ 40 \times 20 \]

\[ 2 \times 20 \]

\[ 2 \times 3 \]
Multiples

Material:
- Bead cabinet and tickets
- The box of colored bead bars
- Paper called *Multiples of Numbers* with numbers 1 to 100 printed in an array
- 3 charts called *Calculation of Multiples* – tables A, B and C, numbered 1 to 50

Presentation 1: Concept and Language of Multiples

1. Lay out the 5 bead chain and skip counting markers; state, “Let’s count the 5 chain. 1, 2, …5 (place the 5 ticket) 6, 7…10 (place the 10 ticket).” etc.
2. Point to and count the numbers in order.
3. State, “Each number contains 5 with nothing left - 5 contains 5 once with nothing left over. Does 10 have 5 in it? (Yes) How many times?” (10 has 2 five’s.) Is there anything left over?” etc.
4. Explain that 5, 10, 15, 20 and 25 are all multiples of 5 and that all numbers that have five in them are multiples of five.
5. State, “When a number contains another number perfectly, it is called a multiple of that number.
6. Repeat with a second chain (6). Use such words as nothing left over, exactly, this many times.
7. Ask the child to state the multiples of the new chain; combine the tickets from both chains and have the child place these in the correct spots.
8. The child can repeat with 2 other chains.

Presentation 2: Concept and Language of Common Multiples

1. Tell the children, “Let’s do multiples of 2.”
2. Take out the bead box. Lay out a two bar horizontally and ask: “How many 2’s are there in two?”
3. Place a two bar vertically below the two bar.
4. Lay out 2 2 bars to the right of the first 2 bar and state, “Two taken two times is 4” Place a four bar underneath the 2 2 bars.
5. Lay out 3 2 bars and ask: “2’s taken 3 times is six?” Place a 6 bar below.
6. Continue by two’s to twelve.
7. State that these are multiples of 2; review how many 2’s are in different numbers.
8. Lay out the 3’s table in the same fashion underneath; have the children find numbers that are in both and introduce as common multiples.
9. Invite the children to do other numbers; 2 digit numbers or 3 numbers rather than just 2 can be used. This work can be written out if the child chooses.
Presentation 3: Investigation of Multiples

1. Introduce the multiples of number paper stating you'll look at multiples using numbers.
2. Choose a number, write it at the top, and circle all it's multiples on the paper.
3. Repeat with a second number and sheet of paper; use a different colored pencil or draw a different shape around each multiple.
4. Later do 2 on the same paper using different colored pencils; the child can also do 3 etc.
5. Large sheets of graph paper can also be used.
6. Children can make booklets of their sheets.

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Presentation 4: Tables A, B and C

1. Table A involves finding multiples of each number (1 – 10) to 50. Bead bars can be used.
2. Table B continues the work of Table A to 100. Counting bead bars may be helpful.
3. On Table C, have the children use tables A and B to fill in the problems that correspond with the numbers on the chart, e.g. for 6, children can write 2x3; 3x2. They can also find prime numbers by asking them to underline in red numbers whose only multiples are 1 and themselves (1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, etc.).
4. Children may also color dots next to multiples on Table C.

Notes:

Multiple (Definition):

When a number contains another number perfectly, it is called a multiple of that number.

‘Multiple’ comes from (Latin) meaning many

Prime Numbers:

Numbers that are only a multiple of one and itself.
### Table A

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### Table B

| 2 x 1 | 2 x 2 | 2 x 3 | 2 x 4 | 2 x 5 | 2 x 6 | 2 x 7 | 2 x 8 | 2 x 9 | 2 x 10 | 2 x 11 | 2 x 12 | 2 x 13 | 2 x 14 | 2 x 15 | 2 x 16 | 2 x 17 | 2 x 18 | 2 x 19 | 2 x 20 | 2 x 21 | 2 x 22 | 2 x 23 | 2 x 24 | 2 x 25 |
|------|------|------|------|------|------|------|------|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 3 x 1 | 3 x 2 | 3 x 3 | 3 x 4 | 3 x 5 | 3 x 6 | 3 x 7 | 3 x 8 | 3 x 9 | 3 x 10 | 3 x 11 | 3 x 12 | 3 x 13 | 3 x 14 | 3 x 15 | 3 x 16 | 3 x 17 | 3 x 18 | 3 x 19 | 3 x 20 | 3 x 21 | 3 x 22 | 3 x 23 | 3 x 24 |

### Chart C

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Lowest Common Multiples

Material:
Pegboard, pegs and small paper strips

Presentation:
1. Place symbols for the numbers 2, 3 and 4 across the top of the pegboard.
2. State, “Today we’re going to look for a special multiple. Let’s look at the multiples of 2,3, and 4.
3. “We’ll use these pegs to help us. We’ll represent each number in pegs.” Place 2 pegs under the 2, 3 under the 3, and 4 under the 4.
4. Tell the rule, “Our only rule for this game is that we add on to the line that has the least first.”
5. Continue to insert pegs below each symbol in 3 columns marking between the multiples of each number with paper strips.
6. Stop when 12 is reached. Acknowledge that they’ve found a common multiple and explain that because it’s the first one found, 12 is called the lowest common multiple of 2,3 and 4.
7. Show how to write: LCM 2, 3, 4 = 12

Notes:
When numbers exceed 10, use the pegs hierarchically, i.e. when 10 green pegs have been counted out, exchange these for one blue peg. (LCM 24, 32 = 96)
Factors

Material:
Pegboard and pegs; paper

Prerequisite: Multiples (and at least an introduction to division.)

Presentation 1: Concept and Language of Factors

1. State, “Let’s try to find the factors or divisors of 18. We will need 18 pegs. We will find the factors of 18 by breaking 18 into groups.” Write 18 on a piece of paper and underline it.

2. Explain, “We can divide 18 by 1 but we are trying to find the factors that are not the number itself or 1. We’ll start with 2.”

3. Divide 18 by 2 on the pegboard by laying out groups of two in a column. “Let’s see if it comes out perfectly.”

4. State, “Two does go evenly into 18 and therefore is a divisor or a factor of 18.” Write “2 yes” under the 18.

5. Ask, “Does 3 go evenly into 18? Let’s try…” Divide 18 pegs into groups of 3: “We were able to make groups of 3 with nothing left over.”

6. “3 is a divisor or factor of 18.” Write “3 yes”.

7. Divide 18 by 4: “18 divided by 4 did not come out perfectly. 4 is not a divisor or factor of 18.” Write “4 no”.

8. Continue with other numbers through 9.

9. Review: 2, 3, 6, and 9 are contained in 18; ask how many times each is contained.

10. Introduce these numbers as factors of 18. Note that wherever you see “yes” on the paper, you can see the factors of the number.

11. Etymology: factor comes from facere (Latin) which means to make; factors are numbers that make a number.

12. Children can find the factors of other numbers; after practice, introduce a number with out factors (prime numbers).

Extension:

Have the child circle the same factors for two numbers and introduce as common factors.
Presentation 2: Concept & Language of Prime Factors Using Table C

**Prerequisite:** Children will have worked with multiples and completed Table C.

1. Discuss what factors are and give an example, e.g. 2 and 3 are factors of 6.
2. Look at Table C, state that the problems written there contain factors of the numbers written before them.
3. Explain that you’ll take one number and look at it more carefully; write the number with it's factors underneath:

   - 18
   - 2x9
   - 3x6
   - 6x3
   - 9x2

4. Look at the first factor set (2 x 9) and determine the factors for 9. Write as: 2 x 9 = 2 x 3 x 3
5. Continue with other sets of factors. Point out that the new factors have no other factors. Ask if they notice anything else – that however they’re arranged, the factors for 18 are the same:

   - 18
   - 2x9 = 2x3x3
   - 3x6 = 3x2x3
   - 6x3 = 2x3x3
   - 9x2 = 3x3x2

6. Introduce these numbers as the *prime factors* of 18 and show how to write this: P.F. 18 = 2 x 3 x 3
7. Have each child do a different number to find it's factors.
Presentation 3: Reducing Numbers to their Prime Factors

Materials: Peg board, gray strips cut into a convenient shape for a container, with cards 0 to 9.

Prerequisite: The child has done some preliminary exercises in divisibility.

1. Explain that you will show how to find prime factors by using the pegboard.
2. Place a number (12) in white at the top of the board.
3. “Let’s do the factors of 12.” Place one blue peg (representing 10) to the left of two green pegs under the “1” in the number 12.
4. Ask if 12 is divisible by 2(always start with 2); (Yes) place a gray strip to the right of the pegs and lay out the answer, 6 pegs, in a horizontal line under the “1” (leave a space); place a “2” ticket to the right of the pegs representing 12 and the gray strip.
5. Ask if the answer 6 is divisible by 2. (Yes) lay out the answer, 3 pegs in a line below the 6 pegs (leave a space); place a “2” ticket to the right of the 6 pegs and the gray strip.
6. Ask if 3 is divisible by 2. (No); ask what about 3? (Yes) lay out the answer, 1 peg, below the 3 pegs (space), and a “3” ticket to the right of the 3 pegs and the gray strip.
7. Write: 2 x 2 x 3, saying “These are the prime factors of 12.”
8. Students can repeat with other numbers.

Presentation 4: Prime Factors & Lowest Common Multiples Together

1. Have students find the prime factors of 3 two-digit numbers.
2. Write the prime factor notation on a piece of paper for each:
   - P.F. 12 = 2x2x3
   - P.F. 24 = 2x2x2x3
   - P.F. 36 = 2x2x3x3
3. Write L.C.M. 12, 24, 36 = ; state that you want to find what the lowest common multiple (L.C.M.) is using the prime factors (P.F).
4. Combine enough factors to make all three numbers, i.e. for 12, you need: 2x2x3; for 24, you need all factors of 12 plus 2; for 36 you need all factors of 12 plus 3.
5. Add all the ones you have to have: 2x2x3 plus 2 plus 3 = 2x2x3x2x3.
6. Put in order and multiply: 2x2x2x3x3 = 72 ; complete notation: L.C.M. 12, 24, 36 = 72
7. This can be checked by working it out on the peg board.
8. After exponential notation have children write in that form, i.e. 2^3 x 3^2.
Notes for Presentation 4:

\[
\begin{align*}
\text{L.C.M.} & \quad 12, 24, 36 = ? \\
2, 2, 2, 3, 3 \\
\Pi(2, 2, 2, 3, 3) & = 72 \\
\text{L.C.M.} & \quad 12, 24, 36 = 72
\end{align*}
\]
Highest Common Factors

Presentation:
1. Have a child work out on the pegboard the prime factors of 2 numbers.
2. Find factors common to both numbers:
   \[
   \begin{align*}
   15 &= 3 \times 5 \\
   27 &= 3 \times 3 \times 3
   \end{align*}
   \]
   \{ 3 \} is the highest common factor (H.C.F.)

   \[
   \begin{align*}
   20 &= 2 \times 2 \times 5 \\
   36 &= 2 \times 2 \times 3 \times 3
   \end{align*}
   \]
   \{ 4 \times 2 \} is the highest common factor (H.C.F.)

3. Introduce notation: H.C.F. 20, 36 = 4; (or can be referred to as the greatest common factor and notated as G.C.F.)
4. Review as the largest number that is a factor of the given numbers (it has to have a factor or group of factors that is the same in all the numbers).
Measurement

Introduction to Measurement

Measurement is the process of finding how many measuring units there are in something. To find out how many there is of something, a devise needs to be used (inches, pounds, gallons, etc.). Measurement pervades everything in our lives; the food we eat, gas, doctors’ prescriptions, the temperature, etc. There are 2 main components of measurement: number and a unit. Either by itself is not a measurement. The number tells how many of the unit are present. To say that someone is 60 tall doesn’t give a measurement or does saying someone is inches tall. This fact should be brought out in the presentation. There are a lot of things to measure, e.g., height, length, weight, time, degrees, dimensions, volume, money, etc. There are also a lot of units of measure that can be taken up by the students for research.

History of Measurement

It is not known from available records where the idea of measurement was first conceived or what the first measurements were. But it seems safe to say that ancient man measured objects and probably used some part of his body as a measuring device. From the records we do have, we know that these civilizations did develop a measurement system based on units, the length of certain parts of the body. In Egypt, around 3,000 B.C., the basic unit was called the cubit, which was the length of a person’s stretched out forearm from the elbow to the tip of the middle finger. This comes from the word cubitum from the Latin word for elbow. This distance varied from person to person and as trade became more prevalent, people fought about this and decided to use the Pharaoh’s cubit as the measurement to use. When everyone uses the same length, it’s called the standard. To make everyone aware of this new standard, they sent out a stick the measurement of the Pharaoh’s cubit and even though the ruler changed on occasion, this system still seemed to work.

The people found that they sometimes needed something that wasn’t as long as the cubit and came up with a new measurement called the palm which was the distance across the hand from the base of the fingers. This was then divided into the finger, one palm then equaling 4 fingers. As they experimented, they found 7 palms equaled a cubit. This marked the beginning of measurements being related to each other.

At this same time in history, the Hebrews lived in Egypt and adapted this system adding their own measurement called the span. This was the distance between the tip of the thumb and the baby finger with the hand stretched out. They also developed the pace which was about 2 cubits long or about one yard in the English system. They even had something called the royal foot which about 2/3 the length of the cubit. The Greek then adapted the system and did a lot with the multiples of the fingers. (Children can research this on their own.)

The Romans continued the system and made some important contributions. They developed the unicia, which was the width of the thumb and has come down to us as the inch. They also had something called the a foot which was equal to 12 uncia and another measurement the distance from the tip of a person’s finger to the end of his/her nose called the yard and equaling about 3 feet.
This system of the Romans is very close to the system of measurement adapted throughout Europe. It was not too long ago in 1855 that the yard, more or less the Roman yard, was standardized. Other measurements were then taken from this with the inch as 1/36 of the yard and the foot at 1/3 of it.

There was also a distance for long measurements, called a mile (1,760 yards). The story comes from a Roman army who had a length called a milliare which was 1,000 paces of the Roman army at a march. This was declared to be the mile and equaled 5,280–feet. This system used throughout Europe was known as the English system.

As early as the 1600’s, people talked about better, easier standards. But not until the 1790’s did a group of French scientists develop a system of measurement which would be unchangeable. This system was adapted by France in 1795 but was not a requirement there until 1840. The measurement unit chosen was called the meter and was based on a measurement of the earth; 1/10,000 the distance from the north pole to the equator along a line of longitude passing through Dunkirk, France and Barcelona, Spain. The word meter came from the Greek word metron meaning measure. The meter became the standard measure and other measurements such as volume, consisting of a cubic decimeter called a gram, came from it. The gram was a measure of weight of a cubic centimeter of water at a temperature where it weights the most (4˚ C). It is said that at the time they used materials that did not expand and contract. (Now use the wave length of a gas.) The metric system was developed to unify the measure system of the world and has mostly succeeded except for the United States and Canada.

Measurement of Length

Exercise One: Introduction to Metric System

Materials: Meter stick

Presentation:
1. Introduce the meter; invite children to feel the meter stick.
2. Explain that it is used to measure how tall, wide, long, deep, etc. things are; all of that is called measuring lengths.
3. Measure the table with the child (how long, wide, tall); show how to line; give measurements in terms of about, a little less or more.
4. Point out centimeter divisions on the stick which give a more precise measurement; measure table again in centimeters.
5. Ask children what they would like to measure; they can also write labels using both the abbreviation or written out (can make charts).
6. Note that just as Romans developed a smaller device; (introduce metric rulers) we have too.
7. Note that just as Romans developed a smaller device; (introduce metric rulers) we have too.
8. When children become more advanced, who how to do scale drawings.
Exercise Two: Introduction to Units of the Metric System

Materials: Decimal board, prepared tickets with names of metric divisions; the root (meter) in black, the prefix in color; their abbreviated eviations on their back sides.

Presentation:
1. Place the ticket with the word metric in the units column on the decimal board; state that in the metric system, the meter is the unit of measure and is king.
2. State that in this kingdom, the meter is divided in the same way; not when it is divided into 10 equal parts, it is called the decimeter (deci for 10).
3. Place the decimeter ticket in the 10ths column on the board; show these divisions on the meter stick.
4. State if the meters taken 10 times, it is called a decameter; place tick in 10’s column.
5. Continued going back and forth (divide by 100, multiply by 100, divide by 1,00, multiply by 1,000) until all the tickets are placed either by you or the students.
6. Can at some point not, that when the prefixes were decided on, scientist used deci, centi and milli, Latin for 10, 100, 1,000 and deca, hecto and killo for the same in Greek.
7. Note when people want to write these out, they us abbreviations (dam = deca).
8. Introduce the abbreviations on the decimal board by flipping the laid out cards over.
9. Note that some are used very often, some hardly ever; present these or have the students discover on their own.
10. Can also note that a couple of bigger or smaller measurements are also used; micrometer for a millionth of a meter or megemeter for million meters.
11. Independent work; student can measure and investigate.
**Exercise 3: Exchanging**

**Presentation:**
1. Ask children if they would like to know how many of one category is in the other, e.g., how many decimeters are in a millimeter.
2. Have them write as a fraction placing the decimeter (dm) as the denominator, the millimeter (mm) as the numerator.
3. Work out on the decimal board in the same way as changing from one category to another, i.e., determine how many millimeters in a millimeter (there is 1). Place a 1 in the mm column. How many centimeters in a millimeter (0.1)? Place a zero until you reach a decimeter for an answer of 0.001.
4. Children can repeat with others.
Other Measurements

Liquid Volume: Introduction through the liter; a series of tickets can be completed on the decimal board in the same manner as the meter. Various containers illustrating liter measurements can be introduced; children can do experiments with these (pouring, comparing, etc.).

Weight: The same series of tickets only in grams can be completed. Have in the class a balance scale and brass weights for the students use (from one gram to a kilogram). Children can weight everything in the class. Note that the prefixes are the same as meter and liter.

Time: Introduce time as presented on a digital clock, e.g., 04:16:36 (hours, minutes, seconds). Review the Babylonians as developing the system based on 60. Show how they would calculate the seconds after noon or midnight like: \((4 \times 60^2) + (16 \times 60^1) + (36 \times 60^0) = 15,396\) seconds after noon for the above time.

We also not use the Latin for large amounts of time, e.g., decade for 10 years, century for 100 years, and millennium for a 1,000 years.

Small amounts of time have been developed by scientist, i.e., millisecond for 1/1,000th of a second, microsecond for 1/1,000,000 of a second and nanoseconds for a billionth of a second (used with computers).

Angles: Given through the story of the Babylonians and geometry work.

Money: This should begin with the child's own system.

Temperature: Starting with Celsius, children can chart into other systems or children can find their own.
# III Multiplication

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notes:
Large Bead Frame

Introduction
The digit and its place indicate number. A 7 may be a seven if it is in the units place, or it may be 70 in the tens place. When children work with the golden bead material, place value is unimportant because the objects themselves record place value. The child need not know that a one with a zero following it can represent a ten bar. He only needs to recognize that the ten bar is a group of ten individual beads. When working with the bead frames, this changes. Color and place gain importance while size and weight are abandoned. It’s the position of the bead that tells the value. The bead frame work represents a dramatic leap toward abstract number manipulation. This is the passage to abstraction. Children face two challenges when attempting multiplication: memorizing the combinations or tables, and manipulating these using place value.

Age:
6 to 8 years old, after the child has some familiarity with multiplication tables.

Materials:
Large bead frame – a wooden frame with 7 horizontal wires each strung with 10 beads. The beads are in hierarchical colors with simple units on the top wire progressing to a million on the bottom wire; corresponding numbers are written on the left side with 1 to 100 on white, 1.000 to 100,000 on gray, a million on black.
Bead frame paper – notation paper divided into two identical columns; across the top in each, the families are indicated with the categories marked in corresponding colors under each. Black diamonds mark the spots for commas.
Presentation 1: Introduction to the Bead Frame
1. Bring out the Bead Frame and discuss what the children notice (colors, spacing, quantities, families, etc.)
2. Say “one unit” and slide one green bead on the top wire from left to right.
3. Move the rest of the green unit beads on the top wire from left to right, counting 1 unit to 10 units.
4. Exchange the 10 green unit beads for a ten by sliding all the green beads right to left, and then sliding one blue bead over on the second wire, saying “one ten. Ten units is the same as one ten”
5. Move the rest of the blue ten beads on the second wire from left to right, counting 1 ten to 10 ten.
6. Exchange the 10 blue ten beads for a hundred by sliding all the blue beads right to left, and then sliding one red bead over on the third wire, saying “one hundred.”
7. Move the rest of the red hundred beads on the third wire from left to right, counting 1 hundred to 10 hundreds.
8. Continue with other categories; relate the large numbers to the wooden hierarchical material.

Presentation 2: Forming Numbers
1. Call out from simple to mixed combinations of numbers and have the children form the numbers on the bead frame. The children can also call out and write their own numbers. Proceed to harder combinations by skipping categories e.g. 10,689.

Presentation 3: Bead Frame Paper
1. The bead frame paper has columns colored to match the wires on the frame. It helps the child to abstract place value.
2. Write digits vertically in the right green column from one to nine, moving a unit green bead across with each number.
3. At ten, move the green beads back, slide over one blue bead, and write ‘1’ in the blue column and ‘0’ in the green column, explaining that this number represents 1 ten and 0 units.
4. Repeat to one hundred.
5. At one hundred, return the blue beads, slide over one red bead, and write ‘1’ in the red column, ‘0’ in the blue column and ‘0’ in the green column. Explain that 100 represents 1 hundred, 0 tens and 0 units.
6. Repeat the above procedure through one million.
7. Form numbers on the frame, have a child read the number from the frame and write it on the paper. Work from simple to mixed combinations, adding zeros later.
8. Allow children to make up numbers for each other, taking turns constructing and recording.

Purpose:
For the children to be able to read and write numbers and to understand the place value in the decimal system.
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Short and Long Multiplication

Materials:
- Large bead frame
- Bead frame paper
- Box of Colored Bead Bars (can be used to demonstrate 3 x 40 = 30 x 4)

Presentation 1: One-Digit Multiplier

1. Give the children a short multiplication equation; record this on the left of the bead frame paper:

   423
   \times 3

2. Show the children how to write the equation broken up on the right side of the bead frame paper. Explain that you'll break up the multiplicand into 3 units, 2 tens (or 20), and 4 hundreds (or 400). Write these out on the bead frame paper:

   3
   20
   400

3. State, “We need to multiply each category by 3 so I'll show this by writing a bracket and the number 3.” Place the bracket and 3 to the right of the broken down problem.

4. Ask, “3 taken 3 times is how much?” Show this on the frame by sliding 3 green unit beads from left to right 3 times or counting to 9 directly.

5. State that you'll need to take 2 tens, 3 times; move 6 blue ten beads across the frame.

6. Ask, “4 hundreds taken 3 times would be what?” Discuss how to show 12 hundreds on the frame by sliding over 2 hundreds and 1 thousand.

7. State that our answer is the amount shown on the frame: 1,269.

8. Later, give the language:

   423 - Multiplicand
   \times 3 - Multiplier
   1269 - Product

9. Children can decorate between problems.
## Presentation 1

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Presentation 2: Two-Digit Multiplier

1. Propose the problem 8437 x 34 and write this on the left side of the bead frame paper; explain that we’ll need to take 8,437 4 times, and then 8,437 30 times.
2. State that you’ll start the same way as in short multiplication by breaking down the multiplicand and multiplying this by the unit digit of the multiplier.
3. Record the broken down multiplicand in the right column; place brackets on the right side of the numbers with the unit digit of the multiplier to its right.
4. Carry out each multiplication on the bead frame, i.e. 7 x 4, 30 x 4, 400 x 4, and 8000 x 4.
5. State, “Now we’ll multiply each digit of the multiplicand by 30.” Record the broken down multiplicand again below the first; place a bracket and the number 30 to the right.
6. Explain that taking 7 30 times would be a lot of work and that we would get the same product if we multiplied 70 3 times.
7. Discuss adding a zero to the end of each digit in the multiplicand and then multiplying this by 3 rather than 30.
8. Cross out the broken down multiplicand taken 30 times, and below this record the multiplicand with a red zero added in the units column. Place a bracket and the number 3 to the right.
9. Carry out each multiplication on the bead frame, i.e. 70 x 3, 300 x 3, 4000 x 3, and 80000 x 3.
10. Record the final answer under the problem on the left side of the bead frame paper.
11. After much practice, introduce a 3-digit multiplier, e.g. 384 x 245. The first category should be worked out, recorded, and the frame cleared; repeat with other categories.

Purpose: Further awareness of place value; bringing to attention multiplication by powers of 10.
Presentation 2

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The Checker Board of Multiplication

Prerequisite:
An introduction to the large bead frame and some ability to read large numbers. This is preparation for geometric multiplication.

Points of Consciousness:
Mark your place with your finger

Materials:
• The checkerboard – a board divided into 36 squares (4 rows) representing category colors, green, blue, and red. Printed numbers corresponding to the categories 1 to 100 million are printed along the bottom edge of the board; values from 1 to 1000 are printed along the right side starting with units at the bottom.
• White multiplicand cards and gray multiplier cards
• Colored bead box
• Pencil and paper

Presentation 1: Introduction to Patterns and Placing Units
1. Display the checkerboard and discuss what strikes the child.
2. Place a unit bead in various places and determine its place value.
3. Explain:
   
   If I put 2 beads in the unit place it is 2.
   If I put 2 beads in the ten place it is 20.
   If I put 2 beads in the hundreds place it is 200.
   If I put 2 beads in the millions place it is 2,000,000.

4. Repeat the above with other bead bars and in different rows to confirm the child’s grasp of the concept.
5. Demonstrate placing and reading 2 bars; continue placing more bars in different rows, sliding, and reading.
6. Discuss the simple, thousands, and millions family.
7. Children can take turns placing and reading numbers, or if they like, they can compose numbers and record them.
Presentation 2: Long Multiplication
1. Propose the problem 5267 x 23 =.
2. Lay the white multiplicand digit cards along the bottom of the checkerboard, and the gray multiplier cards up the right side.
3. Turn over the 2 multiplier card; begin multiplying with units times units (7 x 3) and continue to thousands placing the bead amounts in the corresponding squares.
4. Continue with the tens by turning over both gray cards and placing quantities in the second row above the first quantities starting with the units.
5. Combine beads for each category by sliding down diagonally to the left.
6. Add and exchange in each category to get the answer i.e. 121, 141.

Presentation 3: Multiplication Using Facts
1. Multiply each category in the head and place the beads after exchanging in the head.
2. Slide, exchange, and record the product.

Presentation 4: Long Multiplication with Partial Products
1. Write the problem vertically and then horizontally on paper:
   \[
   \begin{array}{c}
   4375 \\
   \times 25
   \end{array}
   =
   \\]
   \[
   4375
   \]
   \[
   \times 25
   \]
   \[
   21,875
   \]
2. Have the child do the first row on the checker board.
3. Exchange and record the partial product. Explain that this is not the answer because we haven't multiplied 4375 by 20 yet.
   \[
   \begin{array}{c}
   4375 \\
   \times 25
   \end{array}
   =
   \\]
   \[
   21,875
   \]
4. Have the child do the second row on the checker board.
5. Don’t slide yet. Exchange and record the partial product.
6. Slide and exchange. Add the partial products on the board, then add up the problem on paper and compare the two.
   \[
   \begin{array}{c}
   4375 \\
   \times 25
   \end{array}
   =
   \\]
   \[
   21,875
   \]
   \[
   + 87,500
   \]
   \[
   109,375
   \]
7. Explain the placing of the numbers.

Extensions:
Give problems with zero.
Carry over mentally
Multiplication with numbers that gives a square like 111 x 111

Age:
Early 6 to 8.

Purpose:
To teach long multiplication and category multiplication

Note:
If the children came up with an answer that is wrong, they can get frustrated when asked to start again from the top. We can just point out the trouble spot.
Presentation 2

1

2

3
Geometrical Form of Multiplication

Materials:
- Graph paper
- A regular pencil and 3 colored pencils (red, green, blue)
- Ruler

Prerequisite:
- Checkerboard up to multiplication with facts.
- Knows a fair amount of their multiplication facts.

Note:
This is the checkerboard recreated on graph paper

Presentation:
1. Explain that you'll do multiplication by drawing and give the equation $3,432 \times 43 =$
2. Write the equation on the top of the graph paper.
3. Start at the lower right corner of the graph paper; mark it with a point.
4. Move 2 spaces to the left (representing the unit digit of the multiplicand).
5. Continue 3 spaces to the left from that point (representing the 10s) and so on for the other digits of the multiplicand (i.e. 4 then 3).
6. Explain that you'll take the number 43 times; starting at the original point, mark 3 spaces up for the unit; 4 more for the tens.
7. Close off the space by connecting the outermost points (making a rectangle).
8. Mark off the different categories with horizontal and vertical lines.
9. Color in the categories with corresponding colors; Write the multiplication in each space with it's answer.
10. List the answers on paper and add for the final product.

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<td>1200</td>
<td>9000</td>
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<td>1220</td>
<td>16000</td>
<td>120000</td>
<td>147,576</td>
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Purpose:
Preparation for squaring
An analysis of multiplication relative to geometric shape.

Age:
7 or 8
$3,432 \times 43 =$

Answer: 147,576
Flat Bead Frame

Materials:
Flat bead frame or Golden Bead Frame – 9 wires each having 10 golden beads. At the top of the frame are the three families, simple, thousands and millions, color coded white, gray and black and containing the unit, ten and hundred of each family in red. Three dots along the right side colored green, red, blue and dark green represent the multiplier column. There are red zeros along the bottom under each wire.

White paper strips
Multiplier cards

Prerequisite:
Checker board up to multiplication with facts
Know a fair amount of their multiplication facts

Notes:
Work with the flat bead frame is a coming together of previous work with frames. It is the closest thing to doing work abstractly. It can be done parallel to work with the bank game and is an individual lesson. This can go parallel with the bank game. Discuss the characteristics of the frame before introducing how to use the frame.

Presentation 1: Introduction with a Two-Digit Multiplier
1. Give the problem 1345 x 34 = .
2. Place a white strip of paper on the zeros of the bead frame.
3. Have the child write the multiplicand on the paper strip, lining the numerals up with the zeros according to their corresponding category.
4. Place the gray multiplier cards on the corresponding dots along the right side.
5. Start with the units; turn over the tens multiplier card.
6. Multiply each digit of the multiplicand by the units of the multiplier starting with the units, and slide the beads, carrying in your head.
7. Move the multiplicand to the tens place exposing the red units zero and multiply by the tens of the multiplier, in the same manner as above.
8. Read the final answer, i.e. 45,730.
9. Later introduce 3 digit multipliers.
Presentation 2:
Recording Partial Product
(Same procedure as presentation 1.)

1. Lay out the frame. Write out the multiplicand and place it at the bottom of the frame. Put the multiplier cards along the right side of the frame.
2. Have the child write the problem down.
3. Multiply by the units of the multiplier and slide the beads, carrying in your head. Have the child record the partial product.
4. Clear the frame by sliding all the beads back.
5. Move the multiplicand to the tens place and multiply by the tens of the multiplier, in same manner as above.
6. Have the child record the second partial product.
7. Have the child redo the problem on the frame and then add the partial products on paper to correct (these should be the same).

Purpose: To arrive at abstraction.
Bank Game

Introduction:
Elementary children like to work in groups and learn through repetition so we have this activity which lends itself to group work and gives opportunities for much repetition. Another characteristic of this child is that he enjoys large work that takes up a lot of space; this lends itself here also.

This activity includes many age levels of children between the very young who are just learning multiplication to older children up to 11. The younger children can participate because there is no writing involved. The material is totally mechanical and can become abstract.

Three children can play this game: One customer who does the figuring, one banker who moves the cards, and one cashier who simplifies the transactions.

Materials:
Bank game box: labels - cashier, secretary, banker, etc., white card products 1 to 9 million, colored cards multiplicands 1 to 9,000, and gray card multipliers 1 to 9; 1 gray double zero card

Prerequisite:
Addition and multiplication facts; allows children to practice their facts and reinforces what is meant by category multiplication (children can use a multiplication chart if they need help)

Presentation 1: One-Digit Multiplier
1. Have the children set up the cards. Units should be at the right of each layout, moving to the left with each succeeding place.
2. Assign roles to the children – a customer (calculates answers), a banker (retrieves the answer from the bank), and a cashier (adds and exchanges the partial products to obtain the final answer).
3. Propose a problem. e.g. 5873 x 6 =.
4. Choose the multiplicand (colored) and multiplier (gray) cards. Stack the multiplicand cards so that the number is visible. Place the multiplier card to the right.
5. Decompose the multiplicand with units on top, 1000s on the bottom.
6. Place the multiplier card (6) to the right of the 3 unit card; invite the customer to calculate the answer (18) and the banker to retrieve the answer from the “bank”, i.e. the white cards with colored numbers.
7. Place the answer in categories with units on the right across from the cards some distance to the right.
8. Slide the multiplier card down and across from the tens (70); invite the customer to calculate the answer and the banker to retrieve the amount from the bank.
9. Place the answer in category columns to the right of the problem.
10. Continue with each category in the decomposed multiplicand; invite the cashier to add and exchange the categories as needed to compose the answer.
11. Recompose the multiplicand cards and place the answer to the right of the problem.
12. Children can repeat with other problems and change roles if they’d like.
Bank Game Layout

Exchange and Compose Answer:

30000
40000 800
400 20
10
8

35238

Bank Game Layout
Presentation 2: Two-Digit Multiplier

1. Propose a problem. 6784 x 43 =
2. Compose and decompose the multiplicand.
3. Multiply each of the decomposed multiplicand with the unit multiplier and place the answer to the right, as previously presented.
4. Next multiply each category with the tens multiplier. To do this, explain that it can be done by multiplying each of the decomposed multiplicand by 40 or a zero can be added to each category which will then be multiplied by 4 (resulting in the same products).
5. Add the partial products to get the final answer.
6. Later, introduce problems with 3 digit multipliers, and demonstrate writing the problem vertically and recording partial products, e.g.:

   
   \[
   \begin{array}{c}
   \text{6784} \\
   \times 43 \\
   \hline
   \text{20,352} \\
   \text{271,360} \\
   \text{291,712}
   \end{array}
   \]

Purpose:

Allows the children further practice with their facts and is a beginning realization of what is meant by category multiplication.

Note: Notice the two grey zeros after the 70. The multiplier in the problem is 263 (4,573 x 263).
Add Partial Products

Presentation 2
notes:
IV Division

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Group Division with Stamp Game

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Divisibility

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Presentation: Divisibility by Three
Presentation: Divisibility by Four
Presentation: Divisibility by Five
Presentation: Divisibility by Eight
Presentation: Divisibility by Nine
Presentation: Divisibility by Eleven
Divisibility by Twenty-Five

Table of Divisibility
Long Division with Racks and Tubes

Introduction:
This is distributive division as in to share a quantity (dividend) amongst a number of skittles (divisor) such as each unit receives the same amount (quotient). Therefore, the quotient is always shown on the unit board.

Prerequisite:
Children should have worked extensively with the golden beads, particularly division with bows (long division), and multiplication materials before this material is presented. This material requires constant rational thought from the child and can be used with 2 or 3 digit divisors.

Materials:
Racks and tubes consisting of 7 racks, each carrying 10 test tubes of 10 beads in the hierarchical colors (3 white ones for units, tens and hundreds with green, blue and red beads, 3 gray in the same colors, 1 black with only green beads)
7 cups colored as above
3 sets of 9 skittles in green, blue, and red
Division boards each in green, blue, and red, paper
Pencils

Presentation 1:

Materials: Just the Unit Division Board and green beads.

1. Bring out the materials and place them in front of the children. Remind them that in division the answer, or quotient, is what each one (unit) gets and each gets the same. This should be familiar to them from work with the golden beads.
2. Display the test tubes, pointing out that each tube contains 10 beads, each rack represents a different category, and each set of racks represents a different family.
3. Display the cups and lay them out in order. Explain that one green bead in the white cup is a unit. One blue bead in the white cup is the same as 10 green unit beads, and so on.
4. Display the unit division board, pointing out the places for the skittles. Remind the child that the skittles represent the divisor.
5. Give a simple problem. 9 ÷ 3 =
6. Place 9 green unit beads in the green/white unit cup.
7. Place 3 green skittles on the unit board
8. Distribute the 9 beads across the board.
9. Explain that the answer to a division problem is the share of one or what one unit gets.
Racks and Tubes

Presentation 1

\[ 9 \div 3 = 3 \]
Presentation 2: One-Digit Divisor with No Remainder

1. Arrange the material on the rug.
2. Propose the problem: $9764 ÷ 4 = \_ \_ \_ \_ \_ \_ \_ \_ \_ $.
3. Have the child record the equation.
4. Count out skittles for the divisor and place at the top of the board.
5. Compose beads for the dividend in the cups (9 green in green/white cup, 7 red in red/white cup, 6 blue in blue/white cup, and 4 green in green/white cup).
6. State, “In division, we always start with the biggest number.” Place the thousands cup and rack positioned horizontally directly over the unit board leaving the other cups and racks for the dividend off to the right, the racks positioned vertically.
7. Distribute the green thousand beads under each skittle one row at a time forming 2 rows with one left over.
8. Record the share of one (i.e. 2).
9. Clear the board, move both the cup and rack off to the left placing the rack vertically; place any remaining 1000 beads in the 100s cup and turn over the empty 1000 cup.
10. Bring down the 100s cup and rack and place the rack horizontally. Exchange remaining green bead for a tube of red beads (10), explaining that 1 thousand equals 10 hundreds.
11. Distribute all of the red hundred beads under each skittle equally (4 rows).
12. Record the share of one.
13. Clear the board, move both the cup and rack off to the left; place any remaining 100 beads in the 10s cup and turn over the empty 100 cup.
14. Bring down the 10s cup and rack and place the rack horizontally. Exchange each remaining red bead for a tube of blue beads (10), explaining that 1 hundred equals 10 tens.
15. Distribute all of the blue ten beads and place them under the skittles equally (4 rows).
16. Record the answer, clear the board, etc.
17. Repeat the above procedure with the green unit beads.
18. After some experience, introduce problems with remainders and the language:
19. Dividend that which is to be divided; from the Latin dividere which comes from dis meaning two or split up, and vid meaning to separate. (Skt. vidhuh “lonely, solitary,”)
20. Divisor the number that divides the dividend; from the same Latin word.
21. Quotient the number of times one quantity is contained in another; from the Latin quot meaning how much.
22. Remainder that which remains after the removal of a part; Latin remanere – 2 parts re for back, manere – to stay.
9,764 ÷ 4 = 2,441
Presentation 3: Two-Digit Divisor with No Remainder

1. Propose the problem: \(53,328 \div 24 = \).
2. Have the child record the problem. Place the ten board to the left of the unit board.
3. Place 2 blue skittles at the top of the tens board and 4 green skittles at the top of the green board.
4. Compose the dividend by counting out and putting the beads into the appropriate cups. Place the 10,000 rack and cup horizontally in front of the tens board and the 1,000 rack and cup in front of the unit board.
5. Simultaneously distribute row by row the blue 10,000 beads on the blue board with the 1,000 beads on the green board (2 rows on each). Explain that each blue skittle represents ten green skittles - the blue skittle gets ten times more than the green skittles.
6. Record the share of one.
7. Clear the board and move the 10,000s cup and rack to the left, exchange any remaining blue 10,000 beads for 10 green 1,000 beads and turn over the empty 10,000 cup.
8. Place the 1,000s cup and rack over the ten's board.
9. Place the 100s cup and rack over the units board.
10. Distribute the green 1,000 beads on the blue board and the red 100 beads on the green board one row at a time (2 rows).
11. Record the share of one.
12. Clear the board, and move the 1,000s cup and rack to the left, exchange any remaining green 1,000 beads for 10 blue 100 beads and turn over the empty 1,000s cup.
13. Place the 100s cup and rack over the ten's board.
14. Place the 10s cup and rack over the units board.
15. Distribute the red 100 beads on the blue board and the blue ten beads on the green board (2 rows).
16. Record the share of one and repeat procedure.
17. Later, introduce problems with 3-digit divisors.
53,328 + 24 =

Share of One = 2

Share of One = 2

Share of One = 2

Share of One = 2

Share of One = 2

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Presentation 4: One-Digit Divisor with Remainder and Writing Partial Quotients

1. Propose the problem: \(19,928 \div 6\). Have the child record the problem in long division form.
2. Place 6 skittles on the unit board and compose the dividend into the appropriate cups.
3. Attempt to distribute the 1 blue ten thousand bead.
4. Exchange it for 10 one thousand beads and add them to the thousand cup.
5. Move the 10,000 rack and cup to the left, and bring down the 1,000 rack and cup. Distribute all of the green thousand beads equally (3 rows).
6. Record the partial quotient (3) over the 1000s category in the dividend and the amount left in the cup under the 1000s category (1).
7. Clear the board and move the 1000s rack and cup to the left.
8. Exchange the remainder for hundred beads and turn over the empty cup.
9. Bring down the 100s rack and cup. Ask how many there are (19) Record this amount on paper by placing a 9 under the 100s category.
10. Distribute all of the red hundred beads equally (3 rows).
11. Record the partial quotient (3) and the amount left in the cup under the 100s category.
12. Clear the board and move the rack and cup to the left. Exchange the remainder for 10 ten beads.
13. Bring down the 10s rack and cup. Ask how many there are (12) Record this amount on paper by placing a 2 under the 10s category. Distribute all of the blue ten beads equally (2 rows).
14. Record the partial quotient and continue the procedure.
15. Later, introduce the traditional algorithm by having the children record how much in total they used up on the board after writing each partial quotient. Under this, have them place the number left in the cup and the new amount brought down. Show how to place a subtraction sign to the left of the amount used, and a line between this amount and the number illustrating how much was left in the cup.
Presentation 5: Two-Digit Divisor with Remainder and Writing Partial Quotients

1. Propose the problem: \(7,886 \div 35 = \). Have the child record the equation.
2. Place 3 blue skittles on blue board (tens) in the divisor and 5 green skittles (units) on green board.
3. Compose the quantity into the appropriate cups. Bring down the 1000 and 100 racks and cups.
4. Distribute the green 1,000 beads on the blue board with the corresponding number of 100 beads on the green board (2 rows on each).
5. Record the partial quotient, the amount left in the 100s cup, and the amount in the 10s cup.
6. Clear the board, exchange the remaining beads and reposition racks and cup appropriately.
7. Distribute the red 100 beads on the blue board with the corresponding number of 10 beads on the green board (2 rows on each).
8. Repeat the process.
9. Later, introduce the traditional algorithm by adding the step where the amount used on the board is recorded (with a subtraction sign and line placed underneath) before what is left in the cup and the next category is written below.
10. Repeat with larger problems and 3 digit divisors.

\[
\begin{array}{c}
\phantom{0} 2 \, 2 \, 5 \\
\hline
3 \, 5 \left| \begin{array}{c}
7 \, 8 \, 8 \, 6 \\
8 \, 8 \\
1 \, 8 \, 6
\end{array} \right.
\end{array}
\]
Presentation 6: Special Cases

There are three cases in which a divisor may contain a zero. First, there may be a zero in the tens place. Next, there may be a zero in the units place. Finally, there may be zeros in both the tens and units places.

Case 1: Zero in the Middle of Divisor
1. Propose the problem: \( 66,867 \div 207 = \).
2. Have the child write it down.
3. Put the skittles at the top of the boards leaving the 10s board without skittles, and put the beads into the cups. Bring down the 10,000s, 1,000s, and 100s.
4. Lay out the **10,000 beads on the red board** and **100 beads on the green board**.
5. Explain that we put no thousands on the blue board because no skittles are standing for tens.
6. Record, exchange, and clear the board.
7. Put out the 1000 beads on the red board and 10 beads on the blue board. Again, no skittles are standing for ten so nothing is distributed there.
8. Record, exchange and clear the board.
9. Put out the 100 beads on the red board and unit beads on the green board.
10. Record the final answer.

Case 2: Zero at the End of Divisor
1. Propose the problem: \( 78,423 \div 320 = \).
2. Have the child record the problem.
3. Put the skittles at the top of the boards, and put the beads into the cups.
4. Lay out the 10,000 beads on the red board and 1000 beads on the blue board. Explain that we put no hundreds on the green board because no skittles are standing for units.
5. Record, exchange and clear the board.
6. Put out the 1000 beads on the red board and 100 beads on the blue board. Again, no skittles are standing for units.
7. Record, exchange and clear the board.
8. Put out the 100 beads on the red board and 10 beads on the blue board. Set aside the remainder (units are not placed on any board).
9. Record the final answer.

Case 3: Zero in the Middle and End of Divisor
1. Propose the problem: \( 6734 \div 500 = \).
2. Have the child record the problem.
3. Put the skittles at the top of the boards, and put the beads into the cups.
4. Lay out the 1,000 beads on the red board. No skittles are standing for tens or units.
5. Record, exchange and clear the board.
6. Exchange 1000 bead for ten hundred beads and put them out. Set aside the remainder.
7. Record the final answer.
Presentation 6: Special Cases

Age: Later primary to late 8.

Purpose: Long division; preparation for division of fractions.

Note: Another division activity called group division will be the final passage without concrete material.
Group Division with Stamp Game

Introduction:
Rather then deriving the answer from what one unit receives, in group division the divisor is subtracted from the quantity as may times as it takes to be used up. The result in either case, distributive or group, is the same.

Prerequisites:
This lesson should be given at the end of long division when the child is ready to do the work on paper. He should have worked with fractions and have done distributive division.

Materials:
Stamp game material from primary; tickets, strip of paper as a division bar.

Presentation 1: Short Division with a One-Digit Divisor

1. Write a problem on a slip of paper: \(9 \div 3 = \).
2. State that nine is the dividend; have the child lay out 9 stamps randomly under the division bar.
3. Write the divisor on a ticket; place to the left of the stamps below the bar.
4. State that you want to find out how many groups of three are in 9; have the child divide 9 stamps into groups of three. Place stamps in three lined up rows.
5. Write the answer on a new ticket; place it over the stamps and the division bar.
6. State that you have 3 groups of 3, which make how much? 9, the quantity you started with.
7. Repeat with a larger problem e.g. \(2,687 \div 5 = \); have the child lay out the stamps in order for each category at the bottom of the working space.
8. Write the divisor on a ticket; place below and to the left of the division bar.
9. Note that there are no groups of 5 in 2 1000s so you'll need to exchange these for 100s.
10. Sort the 100 stamps into groups; place these in aligned rows on the left below the division bar.
11. Write a ticket for the partial quotient and place this over the 100 stamps and the division bar.
12. Exchange the remaining 100s for 10s. Place these in rows of 5 below and to the right of the 100 stamps.
13. Write a ticket for the partial quotient and place this over the 10 stamps and the division bar.
14. Continue the process with the unit stamps.
15. Place any remaining stamps to the right of the 100 stamps and division bar. Write an “r” on a ticket and the amount of the remainder on another ticket; place both to the right of the quotient tickets.
Presentation 1: 1st problem

Presentation 1: 2nd problem
Presentation 2: Long Division with Two-Digit Divisor

1. Write 3,936 ÷ 32 = .
2. Have a child lay out the dividend in categories and write a divisor on a ticket; place it to the left of the stamps.
3. Ask the child if 32 can go into the thousands.
4. Add 2 100 stamps to the hundreds category to make 1 group.
5. Write the partial quotient 1 on a ticket and place it over 100s category.
6. Look at the 100s and 10s categories; ask how many groups of 32 can be made (2).
7. Place the 100 stamps below the already placed 100 stamps and the 2 10 stamps to the right of these; exchange as needed.
8. Place the answer over the 10s; continue finding groups in the 10s and units categories. Place the answer over the units (3).
9. Check by adding each set of categories or multiplying the quotient and divisor.

\[
\begin{align*}
3200 & \quad \text{or} \quad 123 \times 32 = 3,936 \\
640 & \\
+ 96 & \\
3,936 &
\end{align*}
\]

Presentation 3: Long Division with a 3 Digit Divisor

1. Complete in the same manner as above laying out 3 categories at a time rather than two. E.g. 5648 ÷ 245 =

Presentation 4: Holding Place with Zero

2. Show a placeholder of zero where no category is present.
3. Check by reading from the lay out, adding or multiplying.
Presentation 2

3,936 + 32 =

[Diagram showing the sum of 3,936 and 32 using Montessori materials, including units, tens, hundreds, and thousands.]
Presentation 5: Writing on Paper

1. When children can estimate how many groups, you can estimate on paper.
2. **Begin with a single digit divisor** e.g. $612 \div 5 = $; have the child write the problem in long division form on paper.
3. Begin in the same manner; after writing the first answer ticket, record over the same category on paper. (how many groups were made.)
4. Talk through and record how many were used, how many were left, how many were brought down; bring the next category of stamps down.
5. Continue with the next two categories; record remainder.
6. Child can do later just on paper; help the child to review what he did; can be checked with the stamps.

Presentation 6: Writing on Paper with a 3 Digit Divisor

1. Give problem $6,381 \div 203 = $;
2. Have a child layout the stamps and write a divisor ticket.
3. Ask if they can make 203 out of 6? out of 63? out of 638?
4. Talk through arriving at estimating that 203 would go in 3 times; work out with stamps and on paper
5. Bring down the categories of stamps not used; repeat estimating; record process on paper.
6. Add and record remainder.
Presentation 6

\[ 6,381 \div 203 = \]

\[ \begin{array}{c}
1000 & 1000 & 1000 \\
1000 & 1000 & 1000 \\
1000 & 1000 & 1000 \\
\end{array} \]

\[ \begin{array}{c}
100 & 100 & 100 \\
10 & 10 & 10 \\
10 & 10 & 10 \\
\end{array} \]

\[ 1 \]

\[ \begin{array}{c}
3 \\
1 \\
\end{array} \]

\[ 203 \]

\[ \begin{array}{c}
1000 & 1000 & 1000 \\
1000 & 1000 & 1000 \\
1000 & 1000 & 1000 \\
\end{array} \]

\[ \begin{array}{c}
10 & 10 & 10 \\
10 & 10 & 10 \\
10 & 10 & 10 \\
\end{array} \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ 1 \]

\[ \text{Remainder} = 88 \]

\[ 6,381 \div 203 = 31 \text{ R} 88 \]
Divisibility

Material:
Golden bead materials

Introduction:
This work should be spread over two years starting at 6 1/2. Different lessons can be presented to different groups and students can teach one another. After a presentation, give students time to figure others out on their own. Begin with divisibility by 2. Continue with divisibility by 5, and by 4 before continuing with divisibility by the other numbers.

Presentation: Divisibility by Two

1. State: “When we studied multiples we saw how a larger number could be made. Here we look at large numbers and see how they are divided into smaller numbers.”
2. Lay out golden beads representing the quantity 3,584 explaining that we’ll work with divisibility by 2 first.
3. Ask if the number is divisible by 2.
4. State that you could work it out but that there is a faster way.
5. Divide each category into 2, starting from the thousands and ask if each category is divisible by 2. Exchange as necessary; count the two piles to make sure they’re even.
6. When finished, ask if the two groups are even; record 3,584: yes.
7. Propose adding one unit to the quantity and record 3,585.
8. Ask, “Will the number still be divisible by 2?” Record “no” after the numeral.
9. Add another unit to the quantity and ask the same question.
10. Continue, listing the numbers and the results. Note that every other digit can be divided in 2.
11. Give the rule: A number is divisible by 2 if the units consist of 2, 4, 6, 8 and 0.” Note that these numbers are called even numbers and that if the units are divisible by 2, the number will be as well.
12. Large numbers in the millions can be given to test the rule.
Divisibility by 2

3,584
Presentation: Divisibility by Three

1. Write a 4 digit number, e.g. **9,738**
2. Examine what units are divisible by 3 and determine these to be 3, 6 and 9.
3. State that 10 can be if you take one away, 2 tens, or 20, if 2 are taken away, etc.
4. For 100’s, again if one is taken away from each; the same for 1,000’s, i.e. 999 is.
5. Review what was found and lay out the number in the material.
6. Look at the 1,000’s. State that if one is taken away from each 1000 it’s divisible. Place a unit above each 1,000 to show this.
7. Repeat with 100’s and 10’s. Accumulate all the units including the units category and if this number is divisible by 3, the number is as well.
8. Do a few more examples. Give the rule: If the sum of the digits is divisible by 3, then the number is divisible by 3.

**Notes:** The one you take away is, practically speaking, a stand in for some number - either 9, 99, 999, 9999, etc. which are all divisible by 3.
Divisibility by Three

9,738

One for each thousand block = 9

One for each hundred = 7

9 + 7 + 3 + 8 = 27

27 / 3 = 9

One for each ten bead = 3

= 8

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Presentation: Divisibility by Four
1. Lay out golden beads representing the quantity 816.
2. Divide each category into 4 groups beginning with the 100s, exchanging as necessary.
3. Note that the groups are even. Record 816: yes.
4. Add one unit to the quantity and ask if the quantity is still divisible by 4.
5. Add another unit one at a time to the quantity and ask the same question. Underline in red all those numbers that are divisible by 4.
6. Give the rule: The rule for divisibility by 4 is that the last two digits (tens and units) must be divisible by four (or last 2 digits are multiples of 4) or end in a double zero.

\[
\begin{align*}
412 \div 4 &= 103 \\
92,636 \div 4 &= 23,159 \\
12 \div 4 &= 3 \\
36 \div 4 &= 9
\end{align*}
\]

Presentation: Divisibility by Five
1. Lay out golden beads representing the quantity 125.
2. Divide them into 5 groups, exchanging as necessary. Record 125: yes.
3. Add one unit to the quantity, ask if the quantity is still divisible by 5. Record 126: no.
4. Add another unit to the quantity, ask the same question, and repeat the procedure.
5. Give the rule: A number is divisible by 5 if the end ends in 5 or 0.

Presentation: Divisibility by Eight
1. Repeat the same procedure finding that only 8 in the units, 40 and 80 in the tens, 200, 400, 600, 800, in the 100s, and all the 1000s are divisible by 8.
2. Give the Rule: A number is divisible by 8 if the last 3 digits are divisible by 8 (or last 3 digits are multiples of 8).
3. Have the children test large numbers.

Presentation: Divisibility by Nine
1. Repeat the same procedure above.
2. Give the rule: If the sum of the digits of a number is 9, then the number is divisible by 9.

\[
\begin{align*}
4,329 \div 9 &= 481 \\
7,398 \div 9 &= 822 \\
4 + 3 + 2 + 9 &= 18 \\
7 + 3 + 9 + 8 &= 27 \\
18 \div 9 &= 2 \\
27 \div 9 &= 3
\end{align*}
\]
Divisibility by Five

125

EXCHANGE

if 1 added...

if 2 added...

if 3 added...

if 4 added...

if 5 added...

if 10 added...
Presentation: Divisibility by Eleven

1. The child should have completed divisibility by 3 and 9 first.
2. Determine that no units are divisible by 11.
3. Lay out tens determining that none are divisible, but that if one more is added to each, it would be divisible. Place one unit out for each ten.
4. Note that for 100s, if one is taken away from 100 it would be divisible by 11, if 2 is taken away from 200, etc. (99, 198, 297, etc.).
5. For 1000s, note that they’re divisible by 11 if one unit is added to 1000, 2 to 2000, etc. (e.g. 1001, 2002, 3003).
6. Place a plus sign over the 100s, a minus sign over the 10s and 1000s; state that these are running a surplus or a deficit.
7. Record a number: 2,849. Make a chart with surplus numbers (units and 100s) under a plus sign on the left side, and deficit numbers (10s and 1000s) with a minus sign on the right.
8. Add the numbers in each column; subtract the smaller number from the larger; if the resulting number is divisible by 11, the number is.
9. The child can prove it with the material.
10. Give the Rule: If the difference between two sums of nonconsecutive digits is divisible by 11 or is zero, then the number is divisible by 11.

   \[ 2,849 \div 11 = 259 \]
   \[ 2 + 4 = 6 \]
   \[ 8 + 9 = 17 \]
   \[ 17 - 6 = 11 \]

Divisibility by Twenty-Five

1. Repeat the procedure for determining whether a number is divisible by 5.
2. Rule: A number is divisible by 25 if the last two digit ends in 00, 25, 50 or 75.

Table of Divisibility

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>75</td>
<td>no</td>
<td>yes</td>
<td>etc.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Divisibility by Eleven

2,849

+2

-8

8

9

17

2

4

6

17 - 6 = 11
V Fractions

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Introduction to Quantity, Symbol and Language

Materials:
• Fraction metal insets from whole to tenths
• Small tickets and short strips of black paper
• Pencils

Age: Late primary to early elementary

Presentation 1: Quantity
1. Take out the whole, saying, “This is a unit.”
2. Take out the halves, saying “This is also a unit. Here we have two equal pieces. We call each piece a half.”
3. Take the thirds out and ask, “How many equal pieces is this unit divided into?” Count the pieces, and give the term thirds.
4. Take the quarters out and ask, “How many equal pieces is this unit divided into?” Count the pieces, and give the term fourths.
5. Continue as above through the tenths.
6. Explain that we call each of these a fraction of the whole. Fraction comes from the Latin word frangere, meaning to break.”
7. Continue with second period testing having the child take out, count, write labels for, etc.

Presentation 2: Symbol
1. Remember to tailor the information to the child’s interest. If the child is losing interest, continue at a later time.
   If the child’s interest remains high, continue as follows. (Stop here for the primary child.)
2. Place one half on the table, “We’ve divided this unit in halves, there are two equal pieces.”
3. Write 2 on a ticket and place it below one half circle with a black strip dividing the two.
4. Place 1/3 over a black strip to the right of the 1/2, ask, “How many equal pieces is this unit divided into?”
   Write and place a 3 ticket under the 1/3 and black strip.
5. Continue to the tenths if the child remains interested.
6. If the child is interested, mix the labels and have the child place them below the proper fractions.
Presentation 3: Language

1. **Continue on from presentation 2.**

2. Point to the denominators and state, “The number on the bottom is called the denominator. It comes from the Latin word denomiare, meaning name. This number gives us the name of the family or the fraction family we are talking about.”

3. Point to the half circle with the denominator 2. Explain, “If we are talking about one piece from the halves, we write it this way.” Write 1 on a ticket and place it above the denominator 2 with a black strip separating them, and read it as one half.

4. Have the child write tickets for several different single unit numerator fractions.

5. Pull out two pieces from a fraction (say the fourths). Ask, “What if we have 2 pieces on top? The number on top tell us the number of pieces in the family. Now we have two pieces of the thirds family so we have to change the top number from one to two.”

6. Make a ticket for the fraction, and move the insets to the left.

7. Point at the top number and introduce the term numerator from the Latin word numera, meaning number.

8. Make up fractions for the child to label. Ask about numerator and denominator.

**Extensions:**

1. Prepared tickets, especially for younger children for the child to match with the quantity.

2. Have child write a whole fraction on one ticket.

3. The child may trace and color patterns with the insets, much the way the geometric metal insets are used.

4. The children can make fraction booklets and charts.

5. The circle charts can be left in the room after the lesson as a reference.
notes:
Equivalence

**Materials:** Fraction metal insets, Paper tickets.

**Presentation 1: With Circular Insets**
1. Take out 1 piece of the half frame and place it on the table.
2. Ask the child, “Are there other fraction pieces that can fit perfectly into that space?”
3. Try the thirds, they do not fit.
4. Try the fourths and tell the child, “One half is equivalent to 2 fourths.”
5. Place the fourths to the right of the half on the table.
6. Try the fifths; they do not fit.
7. Try the sixths and tell the child “One half is equivalent to 3 sixths.”
8. Place the sixths to the right of the fourths.
9. Continue until ten.
11. If the child’s interest is intact continue as above with thirds and fourths.
12. Eventually remove more than one piece from an inset and see what other pieces will fit exactly.

**Extensions:**
1. The child may record equivalencies in her notebook.
2. The child may also wish to construct a booklet or an equivalence chart.

**Purpose:** Preparation for reduction of fractions

**Presentation 2: With Divided Triangle and Square Insets**

**Materials:** Square metal insets divided into fractions as rectangles and triangles.

Children can make tickets and do labeling for triangle and square insets as well. They can also work with these in much the same way as presentation one above, emphasizing that the triangles make up the same squares, exchange pieces back and forth.

**Purpose:**
This material allows the child to divide into more than ten parts and to see that quantity does not equal shape.

**Presentation 3: With Constructive Triangles**

**Materials:** Constructive triangle set

**Presentation:**
1. Present in much the same way as above, emphasizing those triangle pieces that can be used to make other shapes.
2. Follow up by having the children make booklets and charts, etc.

**Purpose:**
This material demonstrates that a unit can take any shape. If we take a unit of whatever shape and break it into equal pieces, these are called fractions.
Operations: Simple Cases

Materials:

- Circular metal fraction insets
- Tickets and pencils
- Small green skittles

Note:
The first presentation is given without the label to give isolation of difficulty, concept, then symbol. When the child appears ready, introduce labels.

Presentation 1: Addition with Same Denominators

1. Take out one fourth and state, “Here we have one fourth.”
2. Place two other 1/4s to the right of the first and explain that you want to add all of these.
4. Repeat a number of times using simple numbers with younger children.
5. Place 1/7 on the table and 3/7s to its right. Ask the child what each is; write a ticket for each, place underneath and include operational signs, i.e.:

   \[
   \frac{1}{7} + \frac{3}{7} =
   \]

6. Combine the pieces and move to the right of the equals sign; write the answer and place under the pieces.
7. Repeat, and later give the child a problem where reduction can be done at the end, 1/8 + 3/8 =.
8. Lay out the inset and have the child label the fractions.
9. Have the child slide the insets together and label the answer
10. Explain that there is another step that you can do. “We always write the answer with the smallest number possible. Can you find one piece that is equal to this (4/8)?” The child should pull out 1/2. “
Presentation 1

\[
\begin{align*}
\frac{1}{7} + \frac{3}{7} &= \frac{4}{7} \\
\frac{1}{8} + \frac{3}{8} &= \frac{4}{8} \\
\frac{2}{4} &= \frac{1}{2}
\end{align*}
\]
Presentation 2: Subtraction with Same Denominators

1. Put out the 3/4 insets and have the child take away 1/4.
2. Ask what is left. Have the child reduce the fraction.
3. Introduce writing tickets: place 7/8 on the table and have the child write a ticket for it and place underneath.
4. State that you want to take away 5/8; have the children write a second ticket and place it to the right of the first with operational signs.
5. Remove 5/8 from the 7/8; place the remaining pieces to the right and above the equal sign.
6. Write the answer on a ticket and place under the pieces, and determine if it’s in the lowest terms (i.e. using the least amount of pieces).
7. Exchange 2/8 for 1/4; place to the right with another equal sign and a ticket placed underneath.

Presentation 3: Multiplication by a Whole Number

Note:
Extra fraction pieces are necessary.

1. Take 1/3 two times. Ask the child, “How much do we have?”
2. Have the child choose a fraction (numerator and denominator) and multiplier. Solve the problem, using extra fraction pieces as necessary.
3. Introduce writing mixed numbers.
4. Give the problem 3/8 taken 3 times.
5. Layout 3 sets of 3/8 and demonstrate exchanging 8/8 for one whole; record the answer both ways, i.e. 9/8 and 1 1/8.
Presentation 3

\[
\frac{3}{8} \times 3 = \]

\[
\frac{9}{8} = 1 \frac{1}{8}
\]
Presentation 4: Division by a Whole Number

Additional Materials:
- Skittles

1. “Let’s divide 2/3 by 2.” Have the child put out 2 thirds and 2 skittles.
2. Distribute the insets to each skittle, stating that the answer is what one unit receives. The answer is 1/3.
3. Propose the problem: 6/8 ÷ 3 and have the child layout the insets and 3 skittles
4. Ask the child to label the insets.
5. Distribute the insets to each skittle and state that the answer is what each gets, and ask for the answer.
6. Reduce the answer and label it.
Addition and Subtraction with Different Denominator

Materials:
- Circular fraction insets
- blank paper tickets
- small sheets of plastic (approx. 2”-3” square) marked into different fraction equivalents (1/3's, 1/4's, etc.)
- graph paper, and pencils

Note:
Introduce tickets later.

Prerequisites:
The child must know how different fraction families go together, and have some facility with equivalents and same denominator addition and subtraction.

Presentation 1: Addition
1. Lay out 1/4 and 3/8, stating that you are going to add them, but first they need to have the same denominator.
2. Label the fraction insets and encourage the child to discover that 1/4 and 2/8 are equivalent. Replace 1/4 with 2/8.
3. Slide the pieces together, count and reduce for the answer.
4. Repeat the process. Add 1/2 and 1/6.
5. Lay out and label the insets.
6. Replace 1/2 with 3/6, leaving the tickets the way they stand, and slide the pieces together to the right.
7. Count the pieces for the answer and write it down; place the ticket below.
8. Repeat with 3 addends: 1/4 + 1/4 + 1/3. Encourage the children to combine the fourths and find a denominator within the parameters of the material.
First Part

\[
\frac{1}{4} + \frac{3}{8} = \frac{14}{16}
\]

Second Part

\[
\frac{2}{8} + \frac{3}{8} = \frac{5}{8}
\]

\[
\frac{1}{2} + \frac{1}{6} = \frac{5}{6}
\]
**Presentation 2: Subtraction**

1. Lay out 1/2. Ask how one might take 1/6 away from it.
2. Assist the child in converting 1/2 into 3/6. Replace the 1/2 piece, take 1/6 away, count and reduce for the answer.
3. Repeat, subtracting 2/5 from 1/2.
4. Lay out the 1/2 inset and have the child label it with a ticket below and other tickets for – 2/5 =.
5. Figure out the common family with the child. Exchange the 1/2 for 5/10.
6. Take away 4/10, slide those remaining to the right, count them, and record the answer.
7. Introduce the terms: When adding or subtracting, the terms must have the same denominator (family), they are said to have *common denominators*.

**Presentation 3: Leading to Abstraction**

1. “I’m going to show you a way to change fractions so they have a common denominator.”
2. Put out 2/8 and 1/4 and label it with tickets.
3. Ask the child what they would have to do to make these equivalent (find a common denominator).
4. Write 1/4 = 2/8, reminding the child that she already knows that.
5. Explain that you can multiply the denominator by 2 (4 x 2 = 8)
6. Point out that the fraction will not be equivalent until the numerator is also multiplied by 2. Multiply and label:
   \[
   \begin{align*}
   1 \times 2 &= 2 \\
   4 \times 2 &= 8 \\
   \end{align*}
   \]
7. Explain that the reverse can also be done. “You can go from 4/8 to 1/2 as well.”
8. Ask how to get from 8 to 4 (divide by 2). Remind the child that both the denominator and numerator must be divided by 2 for the fraction to remain equivalent:
   \[
   \begin{align*}
   4 \div 2 &= 2 \\
   8 \div 2 &= 4 \\
   \end{align*}
   \]
9. Have the child solve other problems to ensure that this works on paper.
Presentation 2: 2nd problem

\[
\begin{align*}
\frac{1}{2} & \quad - \quad \frac{2}{5} \\
\frac{5}{10} & \quad - \quad \frac{4}{10} \\
\end{align*}
\]
Presentation 4: Finding Common Denominators beyond the Scope of the Material
1. Propose the problem: $1/3 + 1/4$. Lay out the pieces and write the tickets.
2. State that a common denominator is needed. Show the child the pieces of plastic as units divided into fractions (one in thirds, the other in fourths).
3. Place one on top of the other in such a way so that together they are divided into twelfths.
4. Shade $1/3$ of the one divided into thirds and $1/4$ of the other, lay them back together and count for the answer (remember that one corner is shaded twice and must be counted twice).
5. Write the problem on paper: $4/12 + 3/12 = 7/12$.
6. Demonstrate the same method on graph paper. Shade 9 spaces across and 8 down. Count or multiply block for the common denominator and add $(8/72 + 9/72 = 17/72)$.

Presentation 5: Abstraction Using Common Denominator
1. Write the problem on paper vertically, placing the common denominator to the right.
   e.g. $4/5 + 7/8 =$
2. Discuss how to get from 8 to 40 and from 5 to 40. Talk through the process of multiplying.
3. Repeat the above process with subtraction.

Presentation 6: Applying the Lowest Common Multiple to Find the Common Denominator
[See p.36 for illustration of finding L.C.M.]
1. Propose the problem: $1/4 + 1/6 + 1/8 =$. 
2. Ask the child how we might do it. State that we could multiply all the denominators together to get a common one, but there is another way.
3. Review how the L. C. M. is found on the pegboard, do the same on paper to find the common denominator.
4. Solve the problem in the usual fashion.

Presentation 7: Applying P.F. to Find C.D.
- Work out PF process to get LCM or CD

\[
\begin{align*}
3 \times 3 &= 9 \\
12 \times 3 &= 36 \\
8 \times 2 &= 16 \\
18 \times 2 &= 36 \\
2 \times 12 &= 24 \\
+ 3 \times 12 &= 36 \\
\end{align*}
\]

\[
\begin{array}{c}
4 \\
5 \\
\hline
40 \\
\hline
+ \\
8 \\
\hline
40 \\
\end{array}
\]
Presentation 6: Finding the LCM of 4, 6, 8 on paper for Common Denominator of problem.
Presentation 8: Reducing the Terms First to Find the Common Denominator

1. Reduce the terms first (3/6 = 1/2 etc.), and then find the lowest common denominator to solve the problem.
2. If the fraction is large (42/108), first find the highest common factor (6) and divide numerator and denominator by it to reduce the fraction.
3. Repeat with each fraction, find lowest common denominator and solve as usual.

Notes:
Formulation of the Rule

This is a rule for adding and subtracting with fractions of different denominators. Use the following procedure to find the lowest common denominator.

1. Reduce each term by finding the highest common factor and dividing the numerator and denominator by it.
2. Calculate the lowest common multiple of the reduced denominators.
3. Divide the lowest common multiple by each of the denominators. This is the number by which each term in the fraction must be multiplied. Repeat for all terms in the equation.
4. Perform the operation as usual. Reduce the answer to lowest term.
Multiplication

Presentation 1: Sensorially Multiplying Fractions

Example I
1. Put out the whole piece. Tell the child you'd like to take $1 \times \frac{1}{2}$ times. Ask the child what you should do.
2. Replace the whole with two $\frac{1}{2}$s, take $\frac{1}{2}$ away. State that you have taken $1 \times \frac{1}{2}$ times.

Presentation 1

“I would like to 1, or the whole, $\frac{1}{2}$ times.”

“What should we do?”

Take half of 1 and move it over. Now you have taken $1 \frac{1}{2}$ times.
Example II

$4 \times \frac{2}{3} = \boxed{2\frac{2}{3}}$
**Example II**

1. Have the child take 4 wholes from the extra fraction pieces.
2. Ask him to take four $\frac{2}{3}$ times. Ask him how this might be done.
3. Exchange each whole for 3 thirds. State that the denominator of the multiplier tells you how many to divide the wholes into.
4. Take 2 from each whole and combine them for the answer: $2 \frac{2}{3}$. Ask if the answer is larger or smaller than when you began.

**Example III:**

1. Give another problem. Take $\frac{1}{3} \times \frac{1}{3}$.
2. Have the child write out tickets to compose the problem.
3. Ask the child how this might be accomplished (break the $\frac{1}{3}$ into 3 pieces, $\frac{3}{9}$).
4. Move one piece ($\frac{1}{9}$) to the right, to the answer place.
5. Write $\frac{1}{9}$ on a ticket, placing it to the right of the equal sign, below the $\frac{1}{9}$ piece.
6. Ask if the answer is larger or smaller than when we began.

Taking one out of three (one-third) from three-ninths.

"Is the answer larger or smaller than what we started with?"
Example IV

Taking 2 for every three ninths (or two for every two-thirds)

"Is the answer larger or smaller than what we started with?"
**Example IV:**
2. Put out 2/3, have the child write the tickets and place them below the pieces.
3. Have the child exchange each 2/3 for 6 ninths, move 2 from each set to the right to compose the answer (4/9).
4. Have the child write the answer ticket.
5. Ask the child if it is larger or smaller than what we started with.

**Presentation 2: Writing on Paper**
1. Propose the problem: 1 x 2/3. Write it on paper.
2. Ask the child what he must do to solve it (divide the whole into thirds). Write \((1 ÷ 3)\).
3. Ask the child what he must do next (take 2 of these). Write
   \[1 \times 2/3 = (1 ÷ 3) \times 2 = 2/3.\]
4. State that, in this case, the unit was divided by 3.
5. Continue with the problem: 1/2 x 2/3.
6. Write, while talking it through, 1/2 x 2/3 = (1/2 ÷ 3) x 2 = 2/6 = 1/3. Ask what the unit was divided into (6).
7. Continue with the problem: 1/3 x 2/3. Write, while talking it through, 1/3 x 2/3 = (1/3 ÷ 3) x 2 = 2/9. Ask what the unit was divided into (9).
8. **Note:** In every case the denominator of the answer is the number of parts the multiplicand (unit) was divided into, and the numerator tells how many pieces were taken.
9. Illustrate on graph paper. Ask what the denominators are in 1 x 2/3.
10. Mark off a space 1 square by 3 squares, color in 2 parts, the answer is 2/3.
11. Ask what the denominators are in 1/2 x 2/3. Mark off a 2 x 3 square area, color in 1/2 (3 squares).
12. Mark over 2 of the squares again with X’s. The answer is 2/6 or 1/3 because 2 of the 6 squares are double marked.

**Note:**
After working with the graph paper, the child should come to multiplying numerators and denominators. If not, introduce it.

**Additional Materials:**
- Large division skittles (half, red inside; third, orange inside; fourth, green inside)
Put the skittle back together and count up the fractions pieces (i.e. the four whole in this case) to see what the answer is (i.e. the share of one).

\[ 3 \div \frac{3}{4} = 4 \]
Division

1/2 = Red

1/3 = Orange

1/4 = Green

Large Fraction Skittles

Presentation 1: Dividing a Whole Numbers by Fractions

Example I
1. Propose the problem: 1 ÷ 2/3.
2. Ask what the denominator tells you (how many pieces to divide the whole into).
3. Ask what the numerator tells us (how many pieces to take).
4. Take the skittle divided into 3. Lay out two parts of the skittle. Place the third part off to the side, saying that it will wait in the wings for now.
5. Ask the child what needs to be done to the whole to make it match the skittles (divide it in half). Exchange it for 2 halves.
6. Give each skittle piece a half. State that the answer is what one unit, or whole, gets. To find out, we have to bring the final third from the wings and give it a part, too.
7. Combine the skittle pieces into one, and gather the fraction pieces to arrive at 1 1/2 for the answer.
8. Ask the child if the answer is more or less than the dividend (more).

Example II
2. Place out the dividend (3 wholes). Set up 3/4 of the fourths skittle, placing the fourth in the wings. Write and put out tickets describing the problem.
3. Give each of the skittle pieces set out a whole. State that the answer is what each unit gets, bringing back the fourth skittle piece and giving it a whole as well. Put the skittle back together, counting the wholes to get the answer (4).
4. Label the answer. Ask if the answer is more or less than the dividend (more).
\[
\frac{1}{4} \div \frac{2}{3} = \]

```
\[
\frac{1}{4}
\]
```

Distribute

```
\begin{align*}
\text{Place pieces of skittle back together and} \\
\text{count the share each one gets.}
\end{align*}
```

```
\[
\frac{3}{8}
\]
```

"Is the answer more or less than what we started with?"
Presentation 2: Dividing Fractions by Fractions

Example I
1. Propose the problem: $1/4 \div 2/3$.
2. Have the child write the tickets and put out the dividend ($1/4$ fraction).
3. Have the child put out $2/3$ thirds of the thirds skittle, with a third in the wings.
4. Have the child exchange the $1/4$ for $2/8$, and distribute $1/8$ to each of the skittle pieces.
5. Bring forth the final skittle piece, provide it $1/8$.
6. Recompose the skittle, count the fractions for the answer ($3/8$).
7. Have the child label the answer.
8. Ask if the answer is more or less than the dividend (more).

Example II
1. Propose the problem: $3/4 \div 2/3$.
2. Layout the dividend $3/4$.
3. Set out $2/3$ of the skittle; place third off in the wings.
4. Exchange fourths for eighths. Distribute eighths among the skittle pieces (3 to each).
5. Bring back final skittle, give it $3/8$, recompose the skittle, and gather the fraction pieces to get the answer ($1 1/8$).
6. Write and place the answer ticket. Ask if the answer is larger or smaller than the dividend.
Presentation 3

\[
\frac{2}{3} \div \frac{2}{9} =
\]

Remove two thirds from the frame.

“How many times will \( \frac{2}{9} \) fit into the space left?”

Presentation 4

\[
\frac{3}{16} \div \frac{8}{9} =
\]

To get terms that are divisible by 8, we multiply both terms of the dividend \( \frac{3}{16} \) by 8, getting \( \frac{24}{128} \).

The share of one ends up being 27 (27 of 128ths that is) so the answer is: \( \frac{27}{128} \)
**Presentation 3: Group Division**

1. Propose the problem: \( \frac{2}{3} \div \frac{2}{9} \).
2. Take out the dividend \( \frac{2}{3} \) from the frame.
3. Ask the child how many \( \frac{2}{9} \)'s there are in \( \frac{2}{3} \). State that you can find out by seeing how many \( \frac{2}{9} \)'s will fit in the space left in the frame.
4. Have the child try. He should find that \( \frac{2}{9} \)'s fit into the frame 3 times, so the answer is 3.
5. The child can verify this by drawing the skittles as though doing it sensorially.

**Presentation 4: Division Using Cross Multiplication**

1. Propose the problem: \( \frac{3}{16} \div \frac{8}{9} \).
2. Have the child draw the skittles on paper; 8 in a row, 1 set off.
3. Observe that 3 does not go evenly into 8. Explain that to get terms, which are divisible by 8, we multiply both terms of the dividend by 8, getting \( \frac{24}{128} \).
4. Have the child distribute \( \frac{24}{128} \) to the 8 skittles (each will get 3).
5. Bring back the last skittle, give it 3 as well, count the 128ths, and write the answer \( \frac{27}{128} \).

**Presentation 5: Leading to Abstraction**

Now tell the children what you've just been doing: dividing by the numerator of the divisor and multiplying by the denominator of the divisor. So, to divide a number by a fraction, invert the divisor and multiply.

\[
\frac{1}{3} \div \frac{2}{3} = (\frac{1}{3} \div 2) \times 3 = \frac{1}{3} \times \frac{3}{2} = \frac{1}{2}
\]
Word Problems

Word problems help with linguistic and abstract comprehension. The best situation that can occur for the introduction of the word problem is when a problem comes up naturally. For example, the bus cost \$1.25, I have \$0.75, how much more do I need? Keep your ears open for opportunities to ask the how to figure something out. A newspaper article may inspire a problem to solve, or the teacher may propose a weekly problem on the board. The attitude conveyed should be one of finding out what’s happening rather one of solving a problem.

When helping children solve problems, observe what information is given, identify what you have to find out, and what must be done to reach the answer. Encourage students to make drawings to illustrate the problem and demonstrate ways to write down what is given.

1. Mary and Jane had eaten \( \frac{2}{3} \) of a cake. Then they ate another \( \frac{1}{4} \). How much of the cake is left?

   Given: \( \frac{2}{3} + \frac{1}{4} \) eaten
   Find: How much is left?
   Answer: \( \frac{1}{12} \) cake is left

2. Mary and Jane ate \( \frac{2}{3} \) of a cake again. Then they ate \( \frac{1}{4} \) of what was left. How much cake was not eaten?

   Given: \( \frac{2}{3} \) eaten, then \( \frac{1}{4} \) of leftovers
   \[ 1 - \frac{2}{3} = \frac{1}{3} \quad \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} \]
   Find: The part not eaten
   Answer: \( \frac{3}{12} = \frac{1}{4} \) cake not eaten.

3. A faucet fills up \( \frac{3}{8} \) of a bathtub in 1 hour. How long does it take to fill up the whole tub?

   Given: 1 hour, \( \frac{3}{8} \) full
   \[ 1 - \frac{3}{8} = \frac{5}{8} \text{ left to fill} \]
   Find: How long to fill the tub?
   Answer: 2 \( \frac{2}{3} \) hours or 2:40
   \[ 60 \times \frac{2}{3} = \frac{120}{3} = 40 \]
   or \[ \frac{60}{3} = 20 \]
   \[ 20 \times 8 = 160 = 2 \frac{2}{3} \text{ hours.} \]
4. 5/6 of a pole is 10 meters long. How long is the whole pole?

Given: 5/6 pole = 10 m
Find: Length of the pole
Answer: 12 m

5. Last Monday, at the scout meeting, we served punch. We had 3/4 of a gallon, which was just enough for each scout to have 1 cup. At that meeting only 2/3 of the members attended. This month, the entire troop will be there. How much punch will be needed for the whole troop?

Given: 2/3 troop drank 3/4 gallon
Find: How much for the whole troop?
Answer: 1 1/8 gallon

Notes on the Fraction Charts

In the classroom, two types of charts are used. Those that are impressionistic, like the history and geography charts, and those that are factual, like the nomenclature booklets. Because the following are not impressionistic, or an aid to the imagination, they are not part of the presentation, but may be hung on the wall to aid recall of certain information.

1. Charts 1 and 2
   Fractions as a circular unit divided into pieces from 1 to 10. Children may make square charts.

2. Charts 3, 4 and 5
   Accompanies work in equivalence, examples given.

3. Charts 6 to 11
   Addition charts. 6 gives addition of fractions with same denominator, ending with whole numbers; 7, same denominator, ends with fractions; 8, statement: how to add and subtract a fraction with the same denominator; 9 and 10, addition with different denominators; 11, theory of adding and subtracting with different denominators.
4. **Chart 12**
More adding and subtracting.

5. **Chart 13**
   Subtracting with different denominators.

6. **Charts 14 to 17**
   Multiplication charts. 14 gives fraction times a whole number; 15, same plus rule; 16, combination of fraction times whole number and fraction times fraction; 17, fraction times a fraction plus the rule.

7. **Charts 18 to 21**
   Division charts. 18 gives fraction divided by a whole number; 19, fraction by a whole number plus rule; 20, group division; 21, fraction by a fraction.
## VI Decimal Fractions

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Introduction to Decimal Fractions

Introduction:
The beginning exercises of the decimal fraction work should be introduced by the age of 7.5 years. The work should continue until the exercises have been presented to and internalized by the child. This process takes about 2 to 3 years. (My thought is that some of this work with the materials is better suited for some of the younger children around 6.5-7 years of age (late 1st grade, early 2nd grade)).

Before beginning this work, the child must understand the hierarchy of numbers (that 10 of these equal 1 of those, etc.), multiplication and division by powers of ten (the bank game), and that fractions are divisions of units (the circular inset work).

Materials: Fraction metal insets (whole and tenths), a dish with beads in the three hierarchical colors (green, red and blue), decimal cubes.

Presentation 1: Presentation of Quantity
1. Ask the child to identify the unit circular inset, then place it on the mat.
2. Ask the child what one green bead is (also a unit), then place it under the unit inset.
3. “What do I get if I divide a unit into 10 equal parts?”
4. Place the tenth piece to the right of the unit inset, introduce it as a tenth.
5. Introduce a pale blue cube as a tenth as well, and place it below the tenth inset piece.
6. Replace the unit and tenth in their frames, restating that 10 tenths make a unit.
7. Place a cube onto one tenth of the tenths frame. Have the child place cubes on the other tenths in the frame.
8. Put a green bead on the mat with a blue cube beside it. Ask the child to imagine how many pieces the unit would have if each tenth that made it was divided into ten equal parts.
9. Place a pink cube to the right of the blue cube, and introduce it as a hundredth.
10. Ask the child to imagine how many pieces the unit would have if each hundredth were divided into ten equal parts. Place a pale green cube to the right of the pink one, introducing it as a thousandth.
11. Continue as above with a lighter blue cube for ten thousandth and a light pink cube for a hundred thousandth.
12. Ask the child how many of each category it takes to make the next. Also ask her the names of the parts. Put the cubes away.
13. Place a green bead on the mat, and ask the child what you get if you multiply it by ten.
14. Place a blue bead to the left of the green one. Ask the child what you get if you divide a unit by ten.
15. Place a blue cube to the right of the green bead.
16. Continue as above, alternating multiplication and division and placing the appropriate bead or cube in its place on the mat to a hundred thousand.
17. [Place slips with the numeral symbols above each category. Discuss how the symbols relate to the beads and cubes and how those relate to real quantities. Emphasize the association between the symbol and what it represents.]
Child’s Work:

- Child may write labels using words and symbols for the beads and cubes.
- Child can use large newsprint paper to draw large circles section off the circle (or square) into 100 pieces. If you have 100 pale pink cubes, place one on each section of the 100 piece circle.
- They can make a circle or square sectioned off into 1,000 pieces but this is meticulous work and will only be accepted at the right moment.
Presentation 2: Symbol Linked with Quantity

Materials: Hierarchical beads and cubes, the white (product) cards from the bank game to 100,000, the decimal cards, several small black dots, the decimal propeller and a pencil

1. Place a green bead on the mat. Ask the child what represents ten units. Have him lay out a blue bead.
2. Alternate between multiplication and division, laying out beads and cubes, as in the previous lesson.
3. Bring out the product cards and have the child label the beads, starting with the unit.
4. “Hmm, I wonder if there are any labels for these? (point to decimal cubes) Would you like to see a magic trick?”
5. State that the unit will stay, take the ten card, turn it over, and place one of the small black dots as a decimal point. Introduce as one tenth. Repeat with the 100 and 1000.
6. Ask the child if he would like to do some magic. Have him turn over the cards and place them above the cubes with small black dots appropriately placed. Place the decimal cards out, returning the product cards to their places.
7. Ask the child what happens when you multiply the unit by ten, divide by a hundred, etc. Use the three period lesson format (This is…, Show me ten times that…, What is this…, etc.).
8. Point out that the zero on the left of the tenth's card stands for the unit, and we know the one to the right stands for a tenth because of the decimal point separating it from the zero.
9. Introduce the decimal propeller (or decimal balance), stating that the decimal system is balanced; let the child see how the zeros on the propeller correspond. Spin the propeller on your pencil.

Presentation 3: Formation and Reading of Quantity

Materials: The decimal board, the hierarchical beads and cubes, a small aluminum foil crown, a paper candelabra that matches the spacing of the decimal board, and the decimal and whole number cards.

Passage One
1. Place a green bead in the unit space on the decimal board, announcing that the board is the kingdom of the unit.
2. Place the aluminum foil crown around the green bead. Put out beads and cubes representing tens and tenths, hundreds and hundredths, alternating between multiplication and division, and explaining the relations.
3. Point out the numbers across the top. Place the candelabra on the board, stating the unit supports the whole structure.
4. Remove the beads and cubes, and slide the candelabra up so that the dots representing place value appear as though they are the flames of the candles. Point out that as the numbers grow larger, the flames grow darker.
5. Repeat that the unit holds up the whole kingdom and that the unit is the king of the decimal system.
Presentation 3

[Image of a Montessori material with a colorful abacus and a number card.]
Passage Two

1. Place 3 blue cubes in the tenths’ place. Have the child find the card representing this amount (0.3) and put it next to the board.
2. If there is more than one child, have them place quantities and cards for each other. If not, alternate placing quantities and cards with the child. Use one category at a time.
3. After the child has had some practice, place more than ten cubes in a category, and ask the child to find a card for the amount.
4. When child says they can’t, acknowledge that, encourage her to exchange, and show her how to place the cards.
5. Ask the child how we read a number like this. Discuss different ways it might be read (see note below).
6. Using more than one category, lay out beads and cubes, practice reading the quantities.
7. Dictate numbers to the children.

Note: A number, such as 32.674 can be read in several ways:
   1. Thirty-two and six hundred seventy-four thousandths, or
   2. Thirty-two and six tenths, seven hundredths, and four thousandths,

Extensions:
   1. Have one child put out a quantity with the cards and have another put out a quantity with the beads and cubes. See who has created the greater quantity.
   2. Take a quantity of beads and cubes, have the child take another quantity, see who has more.
Simple Operations

Presentation:

Materials: The decimal board, the hierarchical beads and cubes, skittles, the decimal and whole number cards, paper, and pencils.

Passage One: Addition

1. Form two numbers using the cards; place them over one another, and state that you will add them.
2. Have the child lay out the beads and cubes representing the quantities.
3. Add the numbers, starting with the smallest category, and exchanging as necessary.
4. Lay out answer cards as each category is added. Have the child read the answer when finished.
5. Repeat with different quantities. Have the child choose them. You can use more than two addends, if desired.
6. After the child has had lots of practice, it may be appropriate to introduce writing problems on paper; stress the importance of lining up the decimal points. Moving too quickly to abstraction, however, may compromise the child's conceptual grasp.

Passage 1: Addition
Passage Two: Subtraction

1. Have the child form a minuend with the cards and lay it out on the board.
2. Have another child (or yourself) select a subtrahend with the cards. Take it away from the minuend (as above going from smallest to largest, exchanging as necessary).
3. Have the child lay out cards for the answer. Check it by adding.
4. Repeat with other quantities.
5. Later, demonstrate how problem is written on paper.

Passage Three: Multiplication by a unit

1. Propose the problem: 0.36 x 3. Have the child lay out the cards representing the quantities.
2. Have the child lay out the cubes for the smallest category first (0.06 three times). Then lay out the next place (0.3 three times).
3. Exchange, combine and add for the answer. Place the cards for this to the right of or below the problem, depending on how the child constructed it.
4. Repeat with other quantities.
5. Later show the child how to do the problem with pencil and paper.

Passage Four: Division by a unit

1. Give the problem: 3 ÷ 2
2. Have the child lay out the cards. Lay out three green beads and two green skittles.
3. Place one bead below each skittle. Lead the child to comprehend that he may exchange the remaining unit for 10 tenths.
4. Place out the tenths. Add the tenths and units for the answer. Have the child place out the answer cards.
5. Repeat with 3.4 ÷ 2 and 0.25 ÷ 4. Also, propose 4 ÷ 3, to introduce repeating decimals and how to write them.
6. You may try 0.203 ÷ 3, to introduce repeating later in the answer (0.067666…).
Passage 3: Multiplication

Laying it out

Answer

Passage 4: Division
Multiplication of Decimal Fractions

Materials: Decimal board, beads and cubes, decimal cards, gray multiplier cards, cards with zeros and decimal points (0.0, 0.00, 0.000), graph paper, and pencils

Passage One: With a Multiplier Less Than One

Example I
1. Propose the problem: 0.2 x 0.3. Lay out the multiplicand and the gray multiplier card with a zero card.
2. State, “This problem can’t be done the way it stands with the material. But if we make the multiplier a whole number it can.” Take away the decimal card.
3. “But now the problem is changed. But if I put a zero between the decimal point and the 2, (change the card from 0.2 to 0.02) then I can. I have multiplied one term of the equation and divided the other by the same number. So it cancels out, leaving us with a problem we can use the material to solve.”
4. Verify the child understands what you have done, pointing out that 0.2 x 0.3 = 0.02 x 3.
5. Work the problem with the child on the decimal board.
6. Write the problem in fraction form (2/10 x 3/10). Remind the child that you changed it to 2/100 x 3.
   Have the child solve both problems on graph paper.
7. Finally, verify the answers by multiplying on paper.

Example II
1. Propose the problem 6.36 x 0.5. Lay out the cards, then decompose the multiplicand.
2. Change the multiplier to 5, place it next to 0.06. Change 0.06 to 0.006, and multiply.
3. Move the 5 next to 0.3, change 0.3 to 0.03, and multiply.
4. Move the 5 next to 6, change 6 to 0.6, and multiply.
5. Lay the answer out in cards and recompose the problem to its original form.
6. Verify that the answer makes sense to the child, noting that 1/2 of 6.36 is 3.18.
Example 1

\[ 0.2 \times 0.3 = 0.06 \]

\[ 0.02 \times 3 = 0.06 \]

Example II

\[ 0.006 \times 0.3 = 0.0018 \]

\[ 0.6 \times 5 = 3 \]

\[ 6.36 \times 0.5 = 3.18 \]
Example III
1. Propose the problem: $3.61 \times 0.348$.
2. Lay out the cards, then decompose both numbers, placing the smallest components on top (use zero cards to fully decompose the multiplier).
3. Multiply each digit of the multiplicand by each digit of the multiplier.
4. Move the zero card from the multiplier in front of each multiplicand in turn to have whole number multipliers.
5. Replace the zero cards from each multiplier as you go, moving them off to the side.
6. When finished, recompose the original problem, count and exchange cubes for the answer, and lay out cards representing the answer. [$3.61 \times 0.348 = 1.25628$]

\[
3.61 \times 0.348 = \\
0.01 \times 0.008 = 0.00001 \times 8 = 0.00008 \\
0.6 \times 0.008 = 0.0006 \times 8 = 0.0048 \\
3 \times 0.008 = 0.003 \times 8 = 0.024 \\
0.01 \times 0.04 = 0.0001 \times 4 = 0.0004 \\
0.6 \times 0.04 = 0.006 \times 4 = 0.024 \\
3 \times 0.04 = 0.3 \times 4 = 0.12 \\
0.01 \times 0.3 = 0.001 \times 3 = 0.003 \\
0.6 \times 0.3 = 0.18 \\
3 \times 0.3 = 0.9
\]
Passage Two: Writing Partial Products

1. After the child can do horizontal multiplication on paper, propose a problem written vertically:

$$\begin{array}{c}
0.74 \\
\times 0.32 \\
\end{array}$$

2. Have the child set up the problem in cards, and decompose both numbers.

3. After the first digit of the multiplier has been done (0.0004 x 2 = 0.0008 and 0.007 x 2 = 0.014), show him how to write it on paper, placing each digit of the partial product correctly under the problem:

$$\begin{array}{c}
0.74 \\
\times 0.32 \\
\hline
0.0148 \\
\end{array}$$

4. Multiply the second digit of the multiplier, leaving a space on the board between the cubes of each partial product. Write the partial product.

$$\begin{array}{c}
0.74 \\
\times 0.32 \\
\hline
0.0148 \\
+ 0.2220 \\
\hline
0.2368 \\
\end{array}$$

5. Add to obtain the answer first on the board, then on paper, verify that the answers match. Lay out cards for the answer from the board.
Abstraction of the Rule for Multiplying Decimal Fractions

Materials: Paper and pencil.

Presentation:

1. Propose the problem: 0.3 x 0.15. Ask what .3 is as a fraction and write it after the problem. Do the same for 0.15. Multiply the fractions, then write the answer as a decimal.

   \[0.3 \times 0.15 = \frac{3}{10} \times \frac{15}{100} = \frac{45}{1000} = 0.045\]

2. Propose and solve other problems:

   \[0.4 \times 0.5 = \frac{4}{10} \times \frac{5}{10} = \frac{20}{100} = \frac{2}{10} = 0.2\]

   \[0.106 \times 3 = \frac{106}{1000} \times 3 = \frac{318}{1000} = 0.318\]

   \[0.421 \times 0.223 = \frac{421}{1000} \times \frac{223}{1000} = \frac{93883}{1000000} = 0.93883\]

3. Note that in the first problem, the first number was over 10, the second over 100, so the answer is over 1000 (10 x 100). Repeat with the other problems.

4. Reveal that there is a trick. Encourage the child to identify the trick his or herself.

5. If help is needed, ask how many decimal places are in each number and write it above each number.

6. Repeat with the other problem and ask the child if she notices anything.

7. Point out that the number of decimal places in the answer is the sum of the places in the problem.
Division of the Decimal System

Materials:
Hierarchical beads and cubes, small decimal skittles, decimal place holding disks, pencils, and paper

Presentation:
Example I
1. Propose the problem: 8.6 ÷ 4.3. Review that each category gets 10 times less than the one to the left.
2. Have the child lay out beads and cubes for the dividend and skittles for the divisor. Have him place the quantities below the skittles, exchanging as needed.
3. Read the answer as what each unit receives. 8.6 ÷ 4.3 = 2

Example II
1. Propose the problem: 0.4 ÷ 0.25.
2. Proceed as above, noting that the answer is what the unit gets, or 10 times as much as 1/10 received (answer is 1.6, not 0.16). [The answer is what 1 unit receives. This can be laid out at the end...]
Example III

1. Propose the problem: $0.36 \div 0.027$.

2. Proceed as above, marking both the units and tenths places with disks. The answer is 100 times what the hundredth got ($13.3333\ldots$).

The answer can be laid out after the problem has been “solved” or, that is, after the dividend has been shared among the divisors.
Abstraction of the Rule for Dividing Decimal Fractions

Prerequisites: The child should be able to do division on paper abstractly and have had the sensorial division of fractions.

Materials: Hierarchical beads, cubes and skittles, pencils, and paper

Presentation:
Passage One: Division by a Whole Number

1. Propose the problem: 3.3 ÷ 3 (write the dividend in bracket form) Lay out the beads, cubes and skittles, share out the beads and cubes, exchanging as needed, and arrive at the answer: 1.1
2. Have the child repeat the problem on paper, ignoring the decimal point. Ask the child where the decimal point is in 1.1, and have him put it in the answer.
3. Repeat with other problems until the child sees that a pattern has been established. Ask the child what he sees happening (that the decimal point goes directly above its place in the dividend).
4. Ensure the child understands the concept. Repeat with larger numbers, i.e.: 105.293 ÷ 23
5. Check your answers by multiplying.
Passage Two: Division by a Decimal

1. Propose the problem: \( 4 \div 0.25 \) Have the child solve it using the beads, cubes and skittles.
2. Ask the child if the amounts under each skittle represent the answer (no, it’s 1/10 of the answer). State that if you divided by units you get the answer directly.
3. Multiply 0.25 by 100. Ask if we can now do the problem (no, it has changed).
4. Write the problem as a fraction. State that to do it both sides must be multiplied by the same number. Do so and solve the problem.
5. State the rule: to divide a number by a decimal fraction, first multiply each side by whatever number is required to make the divisor a whole number, then divide as usual.
6. Repeat with other problems to ensure that the child understands the concept.
The Conversion of Common Fractions to Decimal Fractions

Introduction:
This lesson provides a sensorial base for changing between common fractions and decimal fractions. In a decimal fraction, the denominator is always a factor of ten, but this is not written. Rather, it is shown with a decimal point.

Prerequisites:
The child should be aware that there is a connection between common and decimal fractions (i.e.: \(0.1 = \frac{1}{10}\)).

Materials:
The circular fraction insets, the centesimal frame, small paper squares, and pencils

Presentation 1: Sensorial Conversion to Hundredths

1. Introduce the centesimal frame by stating that it is a circle divided into 100 equal parts.
2. Ask the child what each part is worth \((1/100)\). Place the unit inset into the frame, stating that it is the unit.
3. Write and place a slip with its value on it \((1)\). Note that the unit takes up all the spaces in the frame.
4. Explain that when we divide something into 100 parts the denominator is 100. Write this on a slip.
5. State that the numerator tells how many parts of the 100 were taken up. Ask the child how many the unit took up. Add this to the slip \((100/100)\).
6. Remove the unit, place it aside with the slip underneath linked by an equal sign. e.g. \(1 = 100/100\)
7. Place the 1/2 inset to left of the frame. Write 1/2 on a slip and place it below the inset.
8. Place an equal sign to the right of the 1/2. Write another slip with 100 as the denominator and place it to the right of the equal sign. e.g. \(\frac{1}{2} = \frac{50}{100}\)
9. Put the inset into the frame, demonstrating how to properly line it up. Ask the child how many spaces the inset takes up, demonstrating how the frame is read.
10. Write 50 as the numerator on the hundredths slip. State that this looks familiar; ask the child if she knows another way to write this (she may state 5/10, lead her to 0.50 or 0.5).
11. Slide the 1/2 inset and the slips to the right of the unit, and repeat with other fractions. First use one inset at a time, later you may use more than one (2/4, 3/5, etc.).
Presentation 2: Conversion to Any Decimal Number

1. Propose the equation $\frac{1}{4} = \frac{?}{100}$. Talk the child through dividing 100 by 4 and using the answer to multiply the numerator.

$$(100 \div 4) \times 1 = 25 \times 1 = 25$$

2. Convert this to a decimal.

$$(100 \div 4) \times 1 = 25 \times 1 = 25 = 0.25$$

3. Continue: $\frac{3}{4} = \frac{?}{100}$: $(100 \div 4) \times 3 = 75 = 0.75$

$$\frac{5}{8} = \frac{?}{1000}$: $(1000 \div 8) \times 5 = 625 = 0.625$$

4. After the experience, coach the child to state the rule: Divide the denominators and use that answer to multiply the numerator to get the unknown numerator; place the decimal point appropriately.

Extensions:

1. Inexact Fraction Equivalence
2. Demonstrate how to divide the numerator by the denominator and round off, i.e.: $\frac{2}{3} = \frac{?}{10}$
3. Show rounding off to nearest hundredth with $\frac{9}{16} = \frac{?}{100}$.
4. Changing decimal fractions to Common ones
5. Count the number of digits to the right of the decimal point, then place the number over a denominator with that number of zeros, i.e.: $0.36849 = \frac{36849}{100000}$. 
The Decimal Checkerboard

Introduction to the Decimal Checkerboard

Materials:
49 squares of felt measuring 7.5 cm: 12 green, 9 blue, 7 red, 9 light blue, 7 pink and 5 light green

Presentation:
1. Lay a green felt square centered on the table stating that it represents the unit and the unit is king.
2. Ask what you get if you take it ten times (10s), and place a blue square to its left.
3. Ask what you get if you divide by ten (tenths), and place a light blue square to the right of the green.
4. Continue alternating between multiplication and division to 1000.
5. Have the child repeat vertically, multiplying up and dividing down starting at the green unit square.
6. Explain that we now have the the unit and its kingdom, but what about the open areas?
7. Ask what we get if we multiply 10 by 10. Put a red square between the blue ones. Repeat with the other open spaces. Note the pattern, especially 10 x 1/10, 100 x 1/100, etc.
8. Acknowledge that we have some categories laid out. Ask the child if there are any more.
9. The child may draw the decimal board on graph paper, including more categories.
Introduction to the Decimal Checkerboard

Complete by filling in the rest
The Decimal Checkerboard

Materials:
The decimal checkerboard, the bead bars 1 - 9, the gray and white number cards 1 – 10, pencils, and paper

Presentation:

Passage One: Introduction

1. Place a single bead on each square, ask the child what the value of each bead is in its place and what operation is used to get to the next square.
2. Point out that the board is a combination of the regular checker board and the decimal board.
3. Have the child read the categories in diagonals and slide the beads in each category together to the lower right.
4. Have the child exchange the beads for bars and read the resulting number (1,234,567.654321).

![Decimal Checkerboard Image]
Passage Two: Multiplication

1. Propose the problem: 0.427 x 32.
2. Show how to set the cards on the board; the multiplier should be placed so that units times units will give units. (gray multipliers on the left vertically and white multiplicand cards placed on the line between the digits of the multiplier).
3. Multiply by the units of the multiplier (turn the tens card over), proceeding in the same manner as on the regular checkerboard.
4. Multiply by the tens (turn the units card over).
5. Combine the beads diagonally along the bottom and left of the board. Exchange as necessary and read the answer.
6. Repeat with other problems. You may propose a whole number problem to demonstrate that it can be done on this board.

Passage Three: Multiplication writing partial products

1. Propose the problem 0.523 x 24. Write it vertically. Set up the board as in Passage Two.
2. Multiply the units and write the partial product. Repeat with the tens. Encourage the child to carry in his head.
3. Add the partial products on paper. Combine and exchange on the board. Compare the answers.
Passage Four: Multiplying on paper

1. Have the child propose a problem. Have her multiply it as though it was composed of whole numbers on paper, ignoring the decimal points.
2. Have the child estimate where the decimal point should go when she has finished.
3. Check your answer by working the problem on the board.

For General Reference

This shows 98.756 x 32.814
The Effects of Multiplication by Powers of Ten

Materials:
Decimal board, hierarchical cubes, one rack of each color from the racks and cubes, paper and pencils

Presentation:

1. Propose the problem: 23 x 10. Write it down. Have the child work it out by placing 23 beads ten times on the decimal board.
2. Exchange beads until you have a readable answer. Ask the child what occurs when you multiply by ten and write it down (all the units become tens and tens hundreds, etc., and you add a zero to the end of the number).
3. Compare the written answer to the one on the board. Have the child clear the board and propose a second problem: 2.3 x 10.
4. “To solve this problem, we just add a zero, right?” Write answer as 2.30.
5. Have the child work it out on the decimal board. Compare the answers. Wonder aloud “What is going on?”
6. Propose another problem: 0.23 x 10. Ask the child what she thinks the answer will be. Work the problem on the board.
7. When this problem does not work, propose the following: 0.023 x 10. Proceed as above.
8. Talk through what happened: in the first problem, we took the number ten times, and digit moved over a place and the zero filled in for the missing units.
9. Ask the child if she can explain what happened. Talk through this and the others, concentrating on how multiplying by ten moves the decimal place over once to the right.
10. State the rule: When multiplying by ten you move the decimal point one place to the right in the number. When we said just add a zero we weren’t quite right.
11. Propose the problem: 0.42 x 100. Ask the child what she thinks the answer will be.
12. Talk through the hundredths becoming units and the tenths becoming tens (decimal point moves two spaces to the right).
13. Note that when multiplying by a hundred, the decimal point moves two spaces to the right. Continue with 1,000.
14. Coach the child to state a rule. When multiplying by a factor of ten, move the decimal point as many places to the right as there zeros in the multiplier.
The Effects of Division by Powers of Ten

Materials:
Pencils and paper, possibly the decimal board, beads, cubes, and skittles

Presentation:

1. Ask the child what he thinks will happen when we divide a number by ten. Write the problems: 100 ÷ 10; 10 ÷ 10; 1 ÷ 10.
2. Talk the child through the first problem, reminding him of the order of the golden bead material, guiding him to the answers.
3. Ask what happened to the decimal point (it moved one place to the left).
4. Repeat with the other problems, showing how units become tenths, tenths, hundredths, etc.
5. Check answers by sharing the appropriate beads and cubes to 10 skittles.
6. Coach the child to state a rule. When dividing by a factor of ten, move the decimal point as many places to the left as there zeros in the divisor.
The Relative Size of Terms When Multiplying Numbers

Introduction:
The purpose of this lesson is to bring to the child’s attention the relative sizes of the multiplicand, multiplier, and product. A further purpose is to enable them to answer the question “Does this answer make sense?” This lesson may not be necessary; the child may intuit its content.

Materials:
Pencils and paper

Presentation:

Passage One: Fractions Less Than One Times Whole or Mixed Numbers Greater Than One.
1. Propose a few examples, have the child write them down:

   \[ 0.25 \times 4 = 1; \quad 0.5 \times 6 = 3; \quad 0.21 \times 2 = 0.42 \]

2. Ask the child if, while looking at the multiplied numbers and product, they see any relationships.
3. Help the child to see that all the answers are larger than the multiplicand. Ask the child why this is so (the multiplicand is taken some number of times greater then one).
4. Help the child to see that the answer is smaller than the multiplier (because multiplicand is less than one).

Passage Two: Decimal Fractions Less Than One Times Decimal Fractions Less Than One
1. Propose a few examples, have the child write them down:

   \[ 0.25 \times 0.2 = 0.05; \quad 0.3 \times 0.6 = 0.18; \quad 0.4 \times 0.8 = 0.32 \]

2. Help the child to see that the products are smaller than both the multiplicand and multiplier (because you are taking a part of a part).
3. To demonstrate the point, refer to graph paper work.

Passage Three: Whole Number or Mixed Fractions Times the Same
1. Propose a few examples, have the child write them down:

   \[ 3.5 \times 1.5 = 5.25; \quad 6.68 \times 4.1 = 27.388; \quad 3 \times 27 = 81 \]

2. Help the child to see that the products are larger than both the multiplicand and multiplier.
3. Lead child to see that the whole number of the answer is approximately the product of the whole numbers of the problem.
The Relative Size of Terms When Dividing Numbers

Introduction:
The purpose of this lesson is to bring to the child’s attention the relative sizes of the dividend, divisor, and quotient. A further purpose is to enable them to answer the question “Does this answer make sense?” This lesson may not be necessary; the child may intuit its content.

Materials:
The decimal board, blank tickets, black decimal points, and pencils

Presentation:

Passage One: Sensorial
1. Ask the child how many tens 10 units are worth.
2. Repeat from 10 hundreds to 10 hundred thousands, i.e. if you have 10 10s, how many 100s is it worth? etc.
3. Ask the child how many hundredths you have if you have one tenth. Repeat to hundred thousandths.
4. “Knowing this, how many 10s are there in 100,000?” Write 100,000/10.
5. Ask how many 100,000s are in 100,000. Write a ticket saying ‘1’ place it in the 100,000s place on the board.
6. Ask how many 10,000s there are in 100,000. Place a ‘0’ ticket to illustrate ten.
7. Ask how many 1,000 in 100,000, place ticket, and continue to the tens place. Determine that there are 10,000 tens in 100,000.
8. Write the answer next to the problem, and clear the tickets.

100,000/10 = 10,000

9. Ask the child how many hundred thousandths there are in 100,000. Write the problem: 100,000/0.0001.
10. Proceed as above to the hundred-thousandths place (10 billion hundred thousandths in 100,000).
11. Return to the first problem (100,000’s in 100,000). Ask where the decimal point was. Place a black point beside the 1 on the board.
12. Continue with the rest of the problems above, moving the point one space to the right each time.
13. Propose the problem: 1000/10. Ask how many places are between 10 and 1,000. Demonstrate on the board.

Passage Two: Abstraction and Inverse
1. Propose the problem: 100,000/0.0001. Count zeros from left to right, remembering not to count the unit zero twice. Put that number of zeros behind a 1 for the answer.
2. To show the inverse, propose the problem: 10/100,000. Ask the child how many 10 are in 10, place a one in the tens place.
3. Ask how many 10 are in 100, place a zero in the hundreds place, repeat to 100,000. The answer is 0.0001.
4. Continue with other problems. State that when a power of ten is divided by a larger power of ten, the decimal point moves left as many places as there are category differences.

10/100,000 = 0.0001
Squares and Cubes of Numbers

Extension of Work With Short Chains

Exercise One: Extension of the Short Chain Work
Presentation:

Passage One: Making Geometric Figures
Passage Two: Circumscribing One Figure Around Another

Exercise Two: Squares
Presentation:

Passage One: The Concept and Notation of Squares
Passage Two: Numerical Values of Squares

Exercise Three: Cubes
Presentation:

Passage One: Concept and Notation of Cubes
Passage Two: Numerical Value of Cubes
Passage Three: Geometric Shapes with the Cube Chains

Games and Exercises with Squares and Cubes

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Presentation:

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Presentation:

Game IV: Decanomial
Presentation:

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Passage Four: Adjusted Decanomial II
Passage Five: The Decanomial Formula

Game Five: Operations

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Passage Two: Subtraction
Passage Three: Multiplication
Passage Four: Division
Squares and Cubes of Numbers

Extension of Work With Short Chains

These are transitional exercises to be given at the beginning of the year, that introduce the concept of the powers of numbers.

Exercise One: Extension of the Short Chain Work

Materials:
- Bead bars and chains
- felt mat

Presentation:

Passage One: Making Geometric Figures

1. Hold up the three chain stating that it is a short chain composed of three groups of three beads.
2. Arrange the chain into a triangle on the mat, noting that the figure has three sides. Ask the child what the figure is called.
3. Hold up the four chain. Notice that it is composed of four groups of four. Have the child arrange it into a geometric figure and ask what the shape is.
4. Hold up the five chain. Ask the child what can be said about it (5 groups of 5 beads), have her arrange it into a figure on the mat and describe it (pentagon).
5. Repeat with the other chains. Note that no shapes can be formed from the one or two chains. You may wish to introduce point and line as components of figures.
6. The child may, if she likes, write labels for the figures.
Passage Two: Circumscribing One Figure Around Another

1. Form a triangle with the three chain. Circumscribe the four chain square around it.
2. Have the child continue circumscribing with the larger chains in order.
3. Note that the smaller shapes are **inscribed** in the larger, and that the larger are **circumscribed** around the smaller (circum- = circle; scribe = to write).

Exercise Two: Squares

**Materials:**
- Bead chains and squares
- a felt mat
- tickets
- pencils

**Presentation:**

Passage One: The Concept and Notation of Squares

1. Lay the four chain on the mat. Ask how many groups it has.
2. Fold the chain into a square. State that you have taken 4 four times.
3. Write “4” on two tickets, and place one to the right and the second below the square.
4. Ask what shape forms when the chain is folded. State that the chain you’ve just folded is the square of four.
5. Place the four square to the right of the folded chain. State that there is a special way to write this while writing $4^2$ on a ticket so the child can see. Note the small 2 beside the larger 4.
6. Review with the child what you’ve just done ($4 \times 4 = 4^2$). Write equal and multiplication signs on tickets and construct the sentence.
7. Repeat with the other chains. Note that even the short chains are called squares. Ask why this is so (they all fold into squares).
Passage Two
Passage Two: Numerical Values of Squares

1. Place one bead at the top and to the left of the mat. “Here is one.”
2. “If I want to make it a square, I must take 1 one time.” Place a second bead to the right of the first.
3. Write $1 \times 1$ on a ticket and place it to the right of the beads. Review and write $1^2$ on a ticket, placing it to the right.
4. Ask how many beads compose the square of one (1). Write that on a ticket and place it to the right.
5. Fold the two chain under the one chain. Repeat the above procedure. Continue with all the chains and squares to ten.

Follow-up: Children may use prepared tickets, mixing and matching the labels to the squares. They may also use graph paper to go beyond the scope of the material.

Exercise Three: Cubes

Materials:
Bead chains, squares and cubes, a felt mat, tickets, and pencils

Presentation:
Passage One: Concept and Notation of Cubes

1. Present the long chain of four. Lay it across the mat, then fold it into four squares. Note that you have made 4 squares of four, or $4^2$ four times.
2. Write tickets for this, and place them below the squares i.e. $4^2 \times 4$.
3. Stack 4 four squares. Ask the child if they recognize the object. Have the child fetch the four cube and place it to the right of the stacked squares.
4. Reveal that there is a special way to write this ($4^3$). Do so and place the ticket below the cube.
5. State that you can see into a cube from three directions: front, side and top.
6. Review what gets multiplied in $4^2$ ($4 \times 4$). Replace $4^2$ ticket with ones reading $4 \times 4$.
7. Ask the child to read the problem: $4 \times 4 \times 4 = 4^3$. “The reason it is written this way is that there are three ways to see into a cube.”
8. Repeat with the other long chains. Ask the child if it makes sense that the chains are called the cube chains.
Passage Two: Numerical Values of Squares
Passage Two: Numerical Value of Cubes

1. Place one bead (square) at the top left of the mat, stating that you'll take it one time.
2. Place a second bead (cube) to the right of the first. Write and place a ticket reading $1^2 \times 1$ to the right of the beads.
3. “We know that $1^2$ is the same as $1 \times 1$. So we could write $1 \times 1 \times 1$ also.” Write and place the ticket to the right.
4. State that that equals $1^3$, write a ticket, and place it to the right.
5. Ask the child how many beads are in the one cube; write the number on a ticket and place it to the right.
6. Repeat as above with the other squares and cubes to ten. The child may count for values. Have the child write tickets.

**Follow-up:** Children may mix and match the labels. They may also want to go beyond the material with graph paper.

Passage Three: Geometric Shapes with the Cube Chains

1. Help the child form a pentagon from the five cube chain by talking him through what he already knows. Note that each side of the pentagon is made of 5 five bars.
2. Ask the child what is formed when 5 five bars are placed together (five square). Place a five square at each corner of the pentagon.
3. Ask the child what is formed when 5 five squares are placed together (five cube). Place a five cube in the center of the pentagon.
4. The child may wish to repeat this passage with other chains and geometric shapes.

**Note:** This is a consolidation activity bringing together all the concepts and shapes previously introduced.
Games and Exercises with Squares and Cubes

Game I: Total Numerical Values of the Squares from One to Ten

Materials:
- The bead cabinet and a mat

Presentation:

1. Have the child bring one of each bead square to the mat.
2. Have the child stack the squares from largest to smallest.
3. When the stack is completed, ask how many beads are in it. Help the child compose and solve the problem.
4. If the child is interested (or ideally, suggests this on his or her own), stack the cubes and find their value.
5. Ask for the difference in the number of beads between the two stacks.

Game II

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tr>
<td>+7</td>
<td>+5</td>
<td>+3</td>
<td>+1</td>
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<tr>
<td>2+2</td>
<td>2+2</td>
<td>2+2</td>
<td>2+2</td>
</tr>
<tr>
<td>16</td>
<td>21</td>
<td>24</td>
<td>25</td>
</tr>
<tr>
<td>10 = 1^2 + 9</td>
<td>20 = 2^2 + 16</td>
<td>30 = 3^2 + 21</td>
<td>40 = 4^2 + 24</td>
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<tr>
<td>50 = 5^2 + 25</td>
<td>60 = 6^2 + 24</td>
<td>70 = 7^2 + 21</td>
<td>80 = 8^2 + 16</td>
</tr>
<tr>
<td>90 = 9^2 + 9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Game II: Building the Tables

Materials:

- The bead cabinet
- a mat
- pencils and paper

Presentation:

1. Have the child lay out the multiplication tables from one to ten. [Illustrate how to begin by placing a one bar in the far left corner, a two bar to its right horizontally and another below vertically, then fill in with 2 two bars, etc., and allow the child to take over.]

2. Remind the child she has done this before, but that was before she learned about squares.

3. Once laid out, ask the child what she knows about squares and one. Turn the one bead around.

4. Ask about squares and 2. Talk through replacing 2 two bars with 1 two square.

5. Repeat with the rest of the squares. Ask the child what she sees happening. Remind her of the commutative law of addition and multiplication, that the order of the terms in the problem does not matter \((2 \times 1 = 1 \times 2)\).

6. Point to 1 ten bar and say, “Here we have one ten times, how much is that worth?”

7. State that you can change it to something else – a one square plus 9. Record: \(10 = 1^2 + 9\).

8. Repeat with two taken 10 times. Record: \(20 = 2^2 + 16\).

9. Continue with the rest, recording as below. Talk the child through the subtractions indicated on previous page as Game II.
Game III: Power Scales

Materials:
- Pencils and paper

Presentation:

1. Have the child write the squares from 1 to 10 across the top of the paper. Have him write the numerical values of each square below it.

2. Have him then subtract the values (two times) as shown below:

3. Repeat, as below, using the cubes of the numbers:

Note: These are called power scales in math books. They demonstrate that growth is constant at different rates between squares and cubes.
Game IV: Decanomial

Materials:
The bead cabinet, a large mat or small rug, graph paper and pencils

Presentation:
Passage One: Multiplication Table (Decanomial)

1. Lay out the multiplication tables on the rug with the child. Randomly quiz her on the tables as you go.
2. “I see something here, do you?” Give clues until they see the squares in the layout.
3. Point out a square, count its sides, and ask the child to get that square from the bead cabinet.
4. Have the child exchange the bead bars for the square, look for other squares, and exchange those as well. Quiz them periodically on why a particular choice is a square (its sides are equal).
5. When all the squares are exchanged, guide the child to see that all the squares fall on a diagonal line through the middle of the board.
6. Review the squares and their values. The child may wish to draw this on graph paper and highlight the squares.

Passage Two: Adjusted Decanomial

1. Point out that you see that $1 \times 2 = 2 \times 1$.
2. Replace the 2 one beads with a two bar. Guide the child to do the same (replacing shorter bars with longer ones) for the rest of the layout.

Begin with the normal multiplication tables layout. Then begin exchanging beads in this way.
Passage Three: Numerical Decanomial

Materials:
Envelopes labeled “decanomial 1”, etc. to 10, each containing two sets of tickets with the multiples after the square of the number written on them (i.e.: Decanomial 2: 6, 8, 10, 12,…20), one envelope labeled “Squares 1 – 10” containing tickets with those numbers on them -

1. Present the envelopes to the child, reading the labels and placing them on the rug.
2. Lay out the tickets from the “Squares 1 – 10” envelope.
3. Empty the “Decanomial 1” envelope and lay the tickets out in their appropriate places.
4. Have the child help you with the other envelopes. Talk about the multiplication tables as you go.
5. Encourage the child to pick up the tickets at random to place them.
6. Later, the child may empty all the envelopes before placing them, and illustrate the material on graph paper.

<table>
<thead>
<tr>
<th>Contents of Envelopes</th>
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</thead>
<tbody>
<tr>
<td>Decanomial 1 - 1,2,3,4,5,6,7,8,9,10 (2 sets)</td>
</tr>
<tr>
<td>Decanomial 2 - 2,6,8,10,12,14,16,18,20 (2 sets)</td>
</tr>
<tr>
<td>Decanomial 3 - 3,12,15,18,21,24,27,30 (2 sets)</td>
</tr>
<tr>
<td>Decanomial 4 - 4,20,24,28,32,36,40 (2 sets)</td>
</tr>
<tr>
<td>Decanomial 5 - 5,30,35,40,45,50 (2 sets)</td>
</tr>
<tr>
<td>Decanomial 6 - 6,42,48,54,60 (2 sets)</td>
</tr>
<tr>
<td>Decanomial 7 - 7,56,63,70 (2 sets)</td>
</tr>
<tr>
<td>Decanomial 8 - 8,72,80 (2 sets)</td>
</tr>
<tr>
<td>Decanomial 9 - 9,90 (2 sets)</td>
</tr>
<tr>
<td>Decanomial 10 - 10 (2 sets)</td>
</tr>
<tr>
<td>Squares 1- 10: 1,4,9,16,25,36,49,64,,81,100</td>
</tr>
</tbody>
</table>

<table>
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</tr>
</thead>
<tbody>
<tr>
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<td>7 14 21 28 35 42 49 56 63 70</td>
</tr>
<tr>
<td>8 16 24 32 40 48 56 64 72 80</td>
</tr>
<tr>
<td>9 18 27 36 45 54 63 72 81 90</td>
</tr>
<tr>
<td>10 20 30 40 50 60 70 80 90 100</td>
</tr>
</tbody>
</table>
Passage Four: Adjusted Decanomial II

Note: This passage uses the original materials.

1. Lay out the multiplication tables with the bead bars with the child.
2. Point out the squares along the diagonal. Replace them with bead squares. State that you see another way to make squares.
3. Help the child to see that if 2 two bars from the sides are combined, they form a two square.
4. Exchange the bead bars for a square and place it on top the other two square.
5. Continue with other bead bars, exchanging bars for squares and stacking them.
6. Indicate that the bead bars are now stacked in squares and the squares look like their stacks form cubes.
7. Bring out the cubes, compare and exchange them.
8. The child may stack the cubes. Point out that once there was a carpet of beads, and now there is a tower.

In the picture below, notice how the bead bars have been grouped to form squares. With the proper materials, you can then stack them to make the cubes of the numbers.

Begin with Adjusted Decanomial (Passage Two)
Passage Five: The Decanomial Formula

1. Remind the child of the carpet of beads, that it was a square. Ask how you could write the sides down as numbers.
2. Lead the child to see that each side would be \(1 + 2 + 3\) and so on to 10. Remind him that a square is a number times itself, so \((1 + 2 + 3...10)^2\) describes the square.
3. “Let’s figure this out.” Explain that \((1 + 2 + 3...10)^2 = (1 + 2 + 3...10) \times (1 + 2 + 3...10)\).
4. “So, if we do \(1 \times 1\) and add it to \(1 \times 2\) and \(1 \times 3\), all the way to \(10 \times 10\), we will know how many beads make up the whole square carpet.” Do the first row with the child to ensure he understands the process, then let him work the problem. Place a check over the 1 in the second column to show it’s been done before moving on.

\[
\begin{align*}
(1 \times 1) +&(1 \times 2) +(1 \times 3) +(1 \times 4) +(1 \times 5) +(1 \times 6) +(1 \times 7) +(1 \times 8) +(1 \times 9) +(1 \times 10) + \\
(2 \times 1) +&(2 \times 2) +(2 \times 3) +(2 \times 4) +(2 \times 5) +(2 \times 6) +(2 \times 7) +(2 \times 8) +(2 \times 9) +(2 \times 10) + \\
(3 \times 1) +&(3 \times 2) +(3 \times 3) +(3 \times 4) +(3 \times 5) +(3 \times 6) +(3 \times 7) +(3 \times 8) +(3 \times 9) +(3 \times 10) + \\
(4 \times 1) +&\ldots\ldots\ldots\ldots\ldots(10 \times 10) = 3,025
\end{align*}
\]
5. Older children may do the same thing with the letters of the alphabet. Note the same diagonal pattern as in the decanomial.

\[
a^2 + ab + ac + ad + ae + af + ag + ah + ai + aj + \\
b a + b^2 + bc + bd + be + bf + bg + bh + bi + bj + \\
ca + cb + c^2 + cd + ce + cf + cg + ch + ci + cj + \\
da + db + dc + d^2\ldots\ldots\ldots
\]
6. An interesting connection is that the sum of the numbers 1 – 10 cubed alone equals the area of the above square:

\[
13 + 23 + 33 + 43 + 53 + 63 + 73 + 83 + 93 + 103 = ?
\]
\[
1 + 8 + 27 + 64 + 125 + 216 + 343 + 512 + 729 + 1000 = 3025
\]
**Game Five: Operations**

**Prerequisite:** The child must know notation for squares and cubes.

**Materials:**
- The bead cabinet
- Pencils and paper

**Passage One: Addition**
1. Write an addition problem horizontally on paper, i.e.: \(3^2 + 3^2\).
2. Have the child put out 2 three squares. Ask how many beads are in each one (9).
3. Write \(3^2 + 3^2 = 9 + 9\). Do the addition and add for the answer (18).
4. Continue with other similar problems. Encourage the child to come up with her own.

**Passage Two: Subtraction**
1. Write the problem \(5^2 - 2^2 = \). Have the child lay out the five square.
2. With the corner of a paper, cover up \(2^2\), and count the beads left for the answer.
3. Write \(5^2 - 2^2 = 25 - 4 = 21\) on paper.

**Passage Three: Multiplication**
1. Write the problem \(5^2 \times 3 = \). Have the child put out 3 five squares. Ask for an estimate of the answer. The child may count them if he wishes.
2. Write \(5^2 \times 3 = 25 \times 3 = 75\) on paper.
3. If the child is interested suggest \(5^2 \times 3^2 = \). Do this one either on paper or with graph paper.

**Passage Four: Division**
1. Write the problem \(5^3 \div 5 = \). Lay out the five cube and five skittles.
2. Exchange the cube for 5 five squares and distribute them to the skittles. Count the value of one for the answer.
3. Write \(5^3 \div 5 = 125 \div 5 = 25\) on paper.

**Extension:** The child may perform mixed operations such as:
\[(5^3 + 3^2) - 8^2, (6^2 \div 3^2) - 22^2, \text{ or } (3^2 \times 4) + (7^3 \div 7)\]
VIII Squaring & Cubing

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**Transformation of a Square**

**Introduction:**
These activities are designed so that the child works with the square extensively and acquires a deep understanding. Not all of these activities must be presented to all children. The child is generally ready for this work at 8 or 9 years of age.

**Prerequisites:**
The child must know what a square is and understand the distributive law \([a(b+c)=ab+ac]\).

**Materials:**
The bead squares 1 to 10, a jar of colored rubber bands, square dot paper, brackets, a felt mat, tickets, graph paper, and pencils

**Presentation:**

**Passage One: Numerical Expression of a Binomial**

**Part A: With Bead Squares**
1. Place the squares on the mat saying that you’ll be working with the bead squares. Hold up the ten square, ask what it is the square of.
2. “I’m going to make another square out of this.” Count off 4 beads, place a rubber band between the fourth and fifth. Rotate the square 90°, count off 4 beads and place a different colored rubber band to create a four square, a six square and 2 rectangles.
3. Note that you’ve made a four square; ask the child if she sees anything else.
4. Point out the six square and 2 equal rectangles (6x4).
5. Have the child choose and make squares with 2 rubber bands. Note the 2 squares, the rectangles, and their values.

**Part B: Writing**
1. “Let’s write this down. We started with 10².” Write that down.
2. Point out that each side is worth 4 + 6 after division by the rubber bands. “So 10² can also be stated as \((4 + 6)²\).
3. Write \(10² = (4 + 6)²\). Ask what the child sees in the square (four square, six square, two 4 x 6 rectangles).
4. Write \(10² = 4² + 2(4 \times 6) + 6²\). Talk through how this equation describes the square.
5. Ask the child what she expects the number of beads in the square to be (100). Work the problem to see if her estimate is accurate.
\[
\begin{align*}
4² &= 16 \\
2(4 \times 6) &= 48 \\
6² &= 36 \\
10² &= 16 + 48 + 36 \\
100 &= 100
\end{align*}
\]
6. Place a check next to the problem to indicate that it is correct. Repeat the above with other squares the child makes.

**Part C: Square Dot Paper**
1. Introduce the square dot paper. State that you will make a square.
2. Color the circles of the square, starting by counting and coloring the dots on the outside edges and working into the middle.
3. Have the child color the other square in a different color and the rectangles in a third.
4. The child may color other squares. These squares may be glued to a sheet of paper with their mathematical analysis written underneath.
5. The child may use graph paper to do larger squares on graph paper, in the same manner described above.
6. Also, the child may create the square on graph paper, cut it apart, and glue it onto another sheet of paper with the equation written beneath.
Passage Two: Numerical Expression of a Trinomial

1. Hold up the ten square and state that you can make even more squares from it.
2. Count off and mark a three square, then a five square diagonally next to it. Point these out and the two square next to them. Write \(10^2 = (3 + 5 + 2)^2\).
3. Point out the rectangles, rewrite the problem and solve it.

\[
10^2 = 3^2 + 2(3 \times 5) + 5^2 + 2(2 \times 3) + 2(2 \times 5) + 2 \\
100 = 9 + 30 + 25 + 12 + 20 + 4 \\
100 = 100
\]

Continue with other bead squares, square dot paper, and graph paper as detailed in Passage Two above.
Passage Three: Expressing the Binomial Algebraically

Note: This material is suited for those in approximately third grade.

1. “We’re going to do something else with squares.” Mark the ten square into \((3 + 7)^2\). State that you have found their identities numerically.
2. “Instead of calling this ‘3’ (indicating the edge of the three square), we’re going to call it ‘a’.
3. Write this on a ticket and place it above the three square.
4. “Instead of calling this ‘7’, we’re going to call it ‘b’. Write a ticket and place it appropriately.
5. Point out that if the top is ‘a’, the side is also ‘a’; so the square is \(a^2\). Write a ticket and put it on the three square.
6. Reveal that the same is true of the seven square. Write a ticket reading ‘\(b^2\)’, and place it appropriately.
7. Continue with the rectangles, writing and placing tickets for them (\(a \times b\)).
8. Note that you’ve labeled all the beads. Tracing the outside edges of the square, ask the child what you started out with (\(a + b\) on both sides).
9. Write and place a ticket reading \((a + b)^2\) = .
10. Talk through moving the tickets from the square to create the formula next to the \((a + b)^2\) = ticket:
    \[ (a + b)^2 = a^2 + (a \times b) + (a \times b) + b^2 \]
11. Point out that \((a \times b) + (a \times b) = 2(a \times b)\). Write a ticket and exchange it for the others:
    \[ (a + b)^2 = a^2 + 2(a \times b) + b^2 \]
12. Have the child use the formula to solve problems with other factors. The child may also wish to try other bead squares, and working on graph paper.

\[
(a+b)^2 = a^2 + 2(axb) + b^2
\]
Passage Four: Expressing the Trinomial Algebraically

1. Take a ten square and mark off two, three, and five squares with rubber bands. State that you will call the 2 square $a^2$, the three square $b^2$, and the five square $c^2$.

2. Have the children label the squares, then the rectangles, talking through the values of each.

3. Determine the value of the ten square, and write its formula as follows on tickets: $(a + b + c)^2$.

4. Point out that $(a + b + c)^2$ is equal to the labeled square.

5. Lay the equation out in the same manner as the binomial, noting where each value is in the square.

\[ a^2 + 2ab + 2ac + b^2 + 2bc + c^2 \]

6. Have the child repeat the process with other squares, like $4 + 3 + 3$. He may wish to do even larger terms on graph paper.

**Extension:** This work may be used to identify and solve polynomials.
Passing from One Square to Another

Materials:
The bead bars and squares from 1 to 10, graph paper and pencils

Presentation:

Passage One: Passing to a Successive Square Numerically
1. Hold up the four square. “Let's see if we can make this into a five square.”
2. Place the four square on top of the five square. Observe that it takes up some of the space of the five square, but more is needed.
3. As you add the bars and beads, state, “If we add one more four bar to each side and a unit bead for the corner, we'll have the square of five.”
4. Have the child repeat this with successive squares (5, 6, 7…)
5. Show how to write what you’ve done out. “We started out with a four square, and wanted to make it into a five square, so we'll start with $5^2 = 4^2 +$.”
6. Continue, noting that you had $1 \times 4$ twice and the second square was $1^2$.
7. Finish writing the problem, then solve it:

$$5^2 = 4^2 + 2(1 \times 4) + 1^2$$
$$5^2 = 16 + 8 + 1$$
$$25 = 25$$

8. “We’ve turned a four square into a five square, can you do something like that?” Make sure the child writes out her examples.
9. The child may also wish to write her squares out in order.

$$2^2 = 1^2 + 2(1 \times 1) + 1^2 = 4$$
$$3^2 = 2^2 + 2(2 \times 1) + 1^2 = 9$$
$$4^2 = 3^2 + 2(3 \times 1) + 1^2 = 16$$

10. The child may also wish to do this work on graph paper.
Passage Two: Passing to a Non-Successive Square Numerically

1. Hold up the four square, say, “I want to make this into a seven square.”
2. Place the four square on the seven square and count how many four bars are necessary on each side (5).
3. Lay the four bars out. Determine what square will fill the corner (5) and put it out.
4. Allow the child to try. Show her how to write out the equation. She may continue on graph paper, should she so desire.

Ideally, use the bead bars WITHOUT the chains in them.
Passage Three: Changing Algebraically

Note: The child must be experienced and ready for this step.

1. Lay out the six and nine squares. Label the top side of the six square ‘a’, and one side of the nine square ‘b’.
2. “We could count to find what to add to fill out the square. But what if we don’t know the numbers.”
3. Write $b^2 = a^2 + \_$. Note that the area needed to add to $a^2$ would be $b - a$, and that we have that 2 times.
4. Talk through writing it out. Solve the problem to test it.

\[
b^2 = a^2 + 2a(b - a) + (b - a)^2 \\
9^2 = 6^2 + 2(6)(9 - 6) + (9 - 6)^2 \\
9^2 = 36 + 36 + 9 \\
81 = 81
\]

5. Have the child show this with the beads. Repeat the above using different squares.

Passage Three
Squaring a Sum and Expressing it Numerically

Materials:
• Gray and white decimal cards
• Colored number cards
• The bead bars, tickets
• A felt mat
• Printed operation cards
• Paper and pencils

Presentation:

Passage One: Binomial

1. Write 4 + 3 on tickets. “If we wanted to square it, this is how it would be written \([(4 + 3)^2]\).”
2. Lay this out with the bead bars and an addition sign.
3. “We want to square this. What does it mean to square something (take it times itself).”
4. After \((4 + 3)^2\), place tickets reading \(= (4 + 3) \times (4 + 3)\) next to the bead bars, which should also be bracketed.
5. Have the child solve the problem with the bead bars.
6. “Let’s calculate this on paper.” Talk the child through the problem, recording the order the digits were multiplied.

\[
(4 + 3)^2 = (4 + 3) \times (4 + 3) \\
(4 + 3)^2 = 4^2 + (4 \times 3) + (3 \times 4) + 3^2 \\
(4 + 3)^2 = 4^2 + 2(4 \times 3) + 3^2
\]

7. Solve the problem. Ask the child how to get the \((4 + 3)^2\) (add them and multiply).

\[
(4 + 3)^2 = 4^2 + 2(4 \times 3) + 3^2 \\
7^2 = 16 + 24 + 9 = 49
\]

8. Repeat with other problems. The child can do them either in her notebook or on graph paper.

\[
(4 + 3)^2 = (4 + 3)(4 + 3)
\]
Passage Two: Trinomial

1. Write $2 + 3 + 4$. State that you’d like to square it $[(2 + 3 + 4)^2]$.
2. Have the child lay out the beads and tickets, and work the problem in the same way as a binomial.

\[
(2 + 3 + 4)^2
\]

\[
\left( \begin{array}{c}
\bullet \\
\bullet
\end{array} +
\begin{array}{c}
\bullet \\
\bullet
\end{array} +
\begin{array}{c}
\bullet \\
\bullet
\end{array} \right)^2 = (2 + 3 + 4)(2 + 3 + 4)
\]

\[
(2 + 3 + 4)^2 = (2 + 3 + 4) \times (2 + 3 + 4)
\]

\[
(2 + 3 + 4)^2 = 2^2 + (2 \times 3) + (2 \times 4) + (3 \times 2) + 3^2 + (3 \times 4) + (4 \times 2) + (4 \times 3) + 4^2
\]

\[
9^2 = 2^2 + 2(2 \times 3) + 2(2 \times 4) + 3^2 + 2(3 \times 4) + 4^2
\]

\[
9^2 = 4 + 12 + 16 + 9 + 24 + 16 = 81
\]

\[
81 = 81
\]

Passage Three: Polynomial

The child may repeat the above procedure employing more than three numbers. It should be laid out and written in a similar manner.
Squaring a Sum and Expressing It Algebraically

Materials:
Same material as squaring numerically, and perhaps the small moveable alphabet

Presentation:

Passage One: Binomial
1. Have the child lay out the problem \((4 + 3)^2\) with the bead material and gray number cards. State that you will solve the problem, but you’ll use ‘a’ to stand for 4 and ‘b’ to stand for 3. Place tickets for these over the corresponding bead bars.
2. Have the child make and place grayed ‘a’ and ‘b’ tickets above the gray number cards.
3. Have the child solve the problem with the beads. Then have him write and place tickets on top of each section.
4. Have the child rearrange the tickets in the same manner as was done numerically.

\[(a + b)^2 = a^2 + 2ab + b^2\]

5. Repeat using other examples as interest dictates.

Passage Two: Trinomial
1. Repeat the procedure followed for binomials, ending with the statement:

\[(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2\]

Passage Three: Polynomial
1. Repeat as in the preceding passages. You may wish to use the small moveable alphabet to identify the variables.
Squaring a Sum and Expressing It Hierarchically

Materials:
Gray and white decimal cards, colored number cards, the golden bead material (passage one), the peg board and pegs (passage two), tickets, a felt mat, printed operation cards, paper and pencils

Presentation:

Passage One: Binomial
1. “Let’s square 23. What is 23 made up of?” (20 + 3)
2. Lay out the problem with the first part in beads and the second in number cards.
3. Have the child solve the problem as they have done before, but use 100 squares immediately (child should be able to exchange in her head).
4. Have the child write the problem out on paper:

\[(20 + 3)^2 = (20 + 3) \times (20 + 3)\]

\[(20 + 3)^2 = 20^2 + (20 \times 3) + (3 \times 20) + 3^2\]

\[(20 + 3)^2 = 20^2 + 2(20 \times 3) + 3^2\]

\[23^2 = 400 + 160 + 9\]

\[529 = 529\]
Passage Two: Introduction to Binomials and Trinomials with the Hierarchical Pegs
1. Have the child lay out a square of twelve on the peg board. State that this square is too large.
2. Explain that you can make the square smaller. Begin by taking up 10 green pegs and exchanging them for 1 blue peg.
3. Continue to exchange, noting that it is getting smaller. When all the green pegs that can be have been exchanged, ask if the square can be made even smaller.
4. Have the child exchange 10 blue for 1 red peg. Make sure the pegs are arranged in a square.
5. Note that to read the square's value you count along the bottom.
6. On paper, figure out what $12^2$ is. Show that it the same as is out on the peg board.

Passage Three: Large Digit Binomial
1. “Let’s figure out the square of 32.” Lay the problem out along the top of the peg board 3 blue pegs plus 2 green ones, the second part in numbers (be sure that both the pegs and the numbers are enclosed in brackets and the operational signs are in place).
2. Have the child solve the problem, putting the pegs anywhere on the board. After the child has finished help her rearrange the pegs into the square below.
3. Have the child calculate the problem on paper to check the peg work.

\[
(30 + 2)^2 = (30 + 2) \times (30 + 2) \\
(30 + 2)^2 = 30^2 + (30 \times 2) + (2 \times 30) + 2^2 \\
(30 + 2)^2 = 30^2 + 2(30 \times 2) + 2^2 \\
32^2 = 900 + 120 + 4 \\
1024 = 1024
\]
Passage Four: Trinomial with Pegs

Part A

1. Set out the problem \((200 + 30 + 5)^2\) with the multiplicand in pegs and the multiplier in colored cards.
2. Have the child solve the problem, putting out the pegs with the products of the first term across the top, those of the second in the middle and those of the third at the bottom.
3. Review what you have done. Add groups of like quantity together on paper for the answer.
4. Have the child check her work with multiplication, using the formula, and by count the pegs across the bottom of the square.
5. Note that categories may contain more than ten when squared. Also, a smaller quantity numerically may take up more space than a larger number in the square.
Part B
1. “What do you think would happen if we tried to square 302?”
2. Have the child solve the problem as above, using strips of paper to mark the missing categories.
3. The child can also solve trinomials and polynomials on graph paper.

Passage Five: Binomial Using Graph Paper

Note: This passage requires only graph paper and colored pencils

1. “We’re going to solve a binomial equation (23^2). But this time we’ll do it on graph paper.”
2. Have the child rule off 2 then 3 along adjacent edges of a square, starting at the top left corner.
3. Scribe lines separating categories. “Remember 10’s x 10’s equals 100’s.” Color the first square red and write the multiplication inside.
4. Repeat with the other sections, ending with the units in the lower right. The child can then add for the answer.
Extracting the Rules of Squaring a Sum

Note: This exercise can serve as review. The child can be asked to do the whole thing.

Materials:
Gray and white decimal cards, colored number cards, the peg board and pegs, tickets, a felt mat, printed operation cards, paper and pencils

Presentation:

1. “We’re going to square 24.” Have the child set the problem up with pegs and colored number cards.
2. Explain that this time 20 will be ‘a’ and 4 will be ‘b’. Label these on the board.
3. Have the child work the problem out. “Since we’re calling 20 ‘a’, how will we write 20^2?” Have the child place a ticket reading a^2 on the board.
4. Continue with the other multiplications, labeling each.
5. Collect the tickets and lay them out in the usual manner, filling in the operational signs and brackets.

\[(a + b)^2 = a^2 + 2ab + b^2\]

6. Ask if the formula applies only to squares. Have the child express what she has learned in words:
The square of a binomial is equal to the square of the first term plus twice the product of both terms plus the square of the second term.
7. Have the child square 24 using the rule, not the pegs.
    \[24^2 = 20^2 + 2(20 \times 4) + 4^2\]
    \[24^2 = 400 + 160 + 16\]
    \[576 = 576\]
8. Have the child apply the rule to other examples.
9. Extract the rule for the trinomial in the same way:
    \[(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2\]
10. Have the child state the rule:
The square of a trinomial is equal to the square of the first term, plus twice the product of the first and second, plus twice the product of the first and third terms, plus the square of the second term, plus twice the product of the second and third terms, plus the square of the third term.
11. Have the child verify the rule as above. She may use graph paper for larger polynomials.
**Generalizations about Multiplication**

1. Give the child the problem 111². Have her work it out with the pegs, then on paper. Ask her how many of each category she has.
2. Note that this is the same as in the peg lay out. Ask the child what categories can be multiplied together to get each category (i.e.: tens = 1’s x 10’s or 10’s x 1’s; hundreds = 1’s x 100’s, 10’s x 10’s, 1 x100’s; etc.).
3. The child can also try 11 and 1,111. Note that by squaring we can find out how many ways there are to get a specific category.
4. Further, only certain categories multiplied by certain other yield squares, i.e.: 10’s x 10’s.

**Guide Squares**

These should be presented as being representative of what the child has been working on. They are the same type of picture they will always get from working with squares. They can refer to these if they have trouble with a problem.
Passing from One Cube to Another

Note: Not all the squaring work must be completed before cubing begins.

Materials:
The wooden cubing material, a felt mat, paper and pencils

Passage One: Passing to a Successive Cube

Note: When introducing this material, point out that the cubes are a shade darker than the squares. All the squaring exercises need not be completed before beginning cubing.

1. Present the cube of four, state that you will make it into a cube of five.
2. Place the five cube to the left of the four cube.
3. Place 3 four squares—one on the top and one each on two adjacent sides—of the cube.
4. State that it is still not complete. Fill in the edges with unit squares, leaving the corner open for a unit cube.
5. Repeat the above until you are sure the child understands what is happening.
6. State that you went from a four cube to a five cube, write: $5^3 = 4^3 + ?$
7. Have the child disassemble the cube, laying the parts in columns: four squares to the right of the four cube, one squares to the right of that, and the one cube on the end.
8. Talk through what was used and how, and solve the problem on paper.

$$5^3 = 4^3 + 3(4^2 \times 1) + 3(1^2 \times 4) + 1^3$$
$$5^3 = 64 + 48 + 12 + 1$$
$$125 = 125$$
9. Ask what the difference between the two cubes is. Have the child explain that the difference is $125 - 64$, or $48 + 12 + 1$
Passage Two: Passing to a Non-Successive Cube

1. “Let’s go from a four cube to a seven cube.”
2. Ask the child how it should be done. Supervise him laying out the four squares on top of the cube and at two adjacent sides.
3. Help the child to see that he should fill the rest in with three squares and the three cube.
4. Have the child break down the cube in the manner he did in passage one. Review what was used to build the cube.

\[ 7^3 = 4^3 + 3(4^2 \times 3) + 3(3^2 \times 4) + 3^3 \\
= 64 + 144 + 108 + 27 \\
= 343 \]

5. Use the method described in passage one to determine the difference between the cubes.
Cubing a Binomial Numerically

Materials:
The colored bead bars and squares, the cubing material, a felt mat, tickets and pencils

Presentation:

Prerequisite: The child must be able to square a binomial.

Passage One: Starting From the Square

1. “How would we go about squaring five?”
2. Write $5 \times 5 = 5^2$. Ask what must be done to cube it. Write $5^2 \times 5 = 5^3$.
3. “How would we square 3 + 5?” Write $(3 + 5) \times (3 + 5) = (3 + 5)^2$.
4. “How would we cube it?” Write $(3 + 5)^2 \times (3 + 5) = (3 + 5)^3$.
5. Have the child lay out the problem with the sum to be cubed in bead bars, the rest in tickets.
6. Have the child lay out the squared part in beads and write tickets for each section.
7. Change the $(3 + 5)^2$ part of the problem to $(3^2 + (5 \times 3) + (3 \times 5) + 5^2)$ using the tickets written to label each section.
8. “What we really want is the cube. So we’ll bring down the 3 and multiply it first.”
9. Bring down the $3^2$ ticket from the multiplicand and place it to the left of the 3 ticket.
10. Multiply them, placing 2 additional three squares on the one already out.
11. Note that this is a three cube and exchange it.
12. Replace the $3^2$ ticket, and bring down the $5 \times 3$ ticket. Multiply and add two sets of 3 five bars to the one already there.
13. Note that there are 5 three squares in the bead bars. Exchange these for 5 three squares from the cubing material, and place them to the right of the three cube.
14. Return the $5 \times 3$ ticket and bring down the $3 \times 5$ one. Multiply as above, laying out 2 additional sets of 5 three bars to the one there. Notice 5 three squares, exchange, and place them on the adjacent side of the three cube, forming an ‘L’ on the mat.
15. Continue as above with $3 \times 5^2$. Multiply, lay out, exchange for 3 five squares from the cubing material and fill the corner of the figure.
16. “We’ve finished that multiplication. Let’s move on to the 5 now.” Replace the $5^2$ ticket, bring down the 5 ticket from the multiplier and proceed as above, constructing another figure to the right of the first.
17. Review what each part represents (“This cube is the $3^2$ taken 3 times, these squares are $(3 \times 5)$ taken three times…etc.”).
18. Place the second layer on top the first, stating that now you have $(3 + 5)^3$.
19. The child may practice on his own with different cubes.
Passage 1:
Layout of 1st part

\[(3 + 5)^3 \times (3 + 5) = (3 + 5)^3\]

\[
(\begin{array}{c}
\text{3} \\
\text{5}
\end{array})^3 = (3 + 5)^3 \times (3 + 5)
\]

\[
(3^2 + (5 \times 3) + (3 \times 5) + 5) \times (3 + 5) =
\]

\[
\begin{align*}
3^2 & \times 3 \\
5 \times 3 & \times 3 \\
3 \times 5 & \times 3 \\
5^2 & \times 3
\end{align*}
\]
Writing

1. As the child is practicing this, show him how to calculate and write it out on paper.
2. Have the child write tickets for each section as it’s placed, stating what it is (for the three cube, write $3^3$ and lay it on top of the cube, etc.).
3. When both layers are complete, lay the tickets out, placing like terms together:

   \[
   3^3 + 3^2 \times 5 + 5^2 \times 3 + 5^3 \\
   3^2 \times 5 + 5^2 \times 3 \\
   3^2 \times 5 + 5^2 \times 3 
   \]

4. “Do you think there’s a shorter way to write this (combining like terms)?”

   \[
   3^3 + 3(3^2 \times 5) + 3(5^2 \times 3) + 5^3 
   \]

5. Have the child work out the problem.

   \[
   (3 + 5)^3 = 27 + 135 + 225 + 125 \\
   8^3 = 512 
   \]

6. Ask the child how the work may be checked ($8 \times 8 \times 8$). The child can later calculate the problem on paper while manipulating the materials.

Passage Two: Starting from the Cube of the First Term

Note: This lesson may never need to be given—the children will discover this on their own.

1. Propose the problem $(6 + 3)^3$ on paper. Ask what we have to start with. Bring out the six cube.
2. State that you would like to make a cube of $6 + 3$, ask what has to be done.
3. Place 3 six squares on 2 adjacent sides and top of the six cube.
4. Explain that you will fill in the remaining space in the same manner as passing ($6$ three squares and the three cube).
5. Ask the child the cube’s value. Break down the cube and lay it out in columns. Write the values on paper and solve the problem.

   \[
   (6 + 3)^3 = 6^3 + 3(6^2 \times 3) + 3(2 \times 6) + 3^3 \\
   9^3 = 216 + 324 + 162 + 27 = 729 
   \]
The Binomial Cube

Materials:
The binomial cube, a felt mat, tickets and pencils

Presentation:

1. Have the child take out the pieces of the cube and place them randomly on the mat.
2. Place the lid in the corner of the open flaps of the box.
3. Note that the red and blue squares both have a length. Label the red length ‘a’, and the blue length ‘b’, noting that it is next to ‘a’.
4. Have the child write and place labels for each section (a^2, ab, ab, b^2).
5. Ask what the lid design is the square of (a + b). State that you would like the cube of a + b.
6. Move the tickets describing the lid onto the mat.
7. “If we want (a + b)^3 then we multiply that (the tickets from the lid) by (a + b). Lay this out in tickets to create the equation:

\[(a + b)^3 = (a^2 + ab + ab + b^2) \times (a + b)\]

8. “This sentence describes, or makes up, the cube. Let’s start multiplying.” Bring down the first two terms to be multiplied (a^2 x a).
9. Write a ticket for the product (a^3). Have the child find the corresponding piece for this and set the ticket on it.
10. Go on to the next operation (ab x a). have the child write a ticket, find the piece, place the piece to the right of a^3 with the ticket on it.
11. Complete the layer and build the next in the same manner, setting the second to the right of the first.
12. Note that all of this together is the cube (a + b)^3. Take the tickets off and combine the layers to make the cube.
13. Lay the tickets out in order, with like terms together, then combine them to form:

\[(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]

14. Ask how many pieces there are in the cube (8), and how many terms. Ask what the cube of two is. Divulge that the child may wish to write this in her notebook.
\[(a+b)^3 = (a^2 + ab + ab + b^2) \times (a + b)\]

\[(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\]
Cubing a Binomial Algebraically

**Materials:**
The binomial cube, a felt mat, tickets and pencils

**Presentation:**

1. After construction of the cube, have the child separate the material into categories, write and place tickets for the pieces, using 6 for the ‘a’ length and 3 for the ‘b’ length.

2. “These tickets describe, or make up, \((6 + 3)^3\). Have the child collect and lay out the tickets:

   \[
   (6 + 3)^3 = 6^3 + (6^2 \times 3) + (6 \times 3^2) + 3^3
   \]
   \[
   (6^2 \times 3) + (6 \times 3^2)
   \]

3. Have the child combine terms and exchange them to create:

   \[
   (6 + 3)^3 = 6^3 + 3(6^2 \times 3) + 3(6 \times 3^2) + 3^3
   \]

4. Calculate the answer. Check it by adding 6 and 3, and cubing them

5. Coach the child to make the statement:

   The cube of a binomial is equal to the cube of the first term, plus three times the first term squared times the second term, plus three times the first term times the second term squared, plus the second term cubed.
Cubing a Trinomial Algebraically

Materials:
The trinomial cube, a felt mat, tickets, prepared tickets, and pencils

Presentation:

1. Place the lid in the space left by the open flaps. Have the child take out the pieces, referring to the lid, and place them on the mat. Label the red length ‘a’, the blue length ‘b’ and the yellow length ‘c’.
2. Have the child label the squares using the variables above. Continue with the rectangles.
3. “What is this equal to (how would we write this)?” \((a + b + c)^2\) “Now, let’s cube it.” Lay out the tickets already written and write tickets to fill out the rest.

\[(a + b + c)^3 = (a^2 + ab + ac + ac + b^2 + bc + bc + c^2) x (a + b + c)\]

4. Work through the problem in the same manner as used with the binomial cube: multiply each term of the multiplicand by each term of the multiplier, locate the piece, label and place it in formation by layers.
5. When finished, remove the tickets and lay them in order, placing like terms in rows.
6. Put the cube together. Write new tickets combining like terms and reconstruct the formula:

\[(a + b + c)^3 = a^3 + 3a^2b + 3ab^2 + 3ac^2 + 6abc + b^3 + 3b^2c + 3bc^2 + c^3\]

7. Have the child practice with the prepared tickets. Note that the lengths were labeled ‘a’, ‘b’, and ‘c’. When an equation has one term, it’s called a monomial. Hold up the red cube, and call it the cube of ‘a’.
8. “If there is a second length, called ‘b’, how would the cube look?” Have the child construct the cube of \((a + b)\) using whatever he sees fit.
9. “Since this cube has two terms, it is called a binomial. What if there was a third length called ‘c’, how would the cube look?” Have the child construct the trinomial cube.
10. “This cube has three terms, it is called a trinomial.” Review the terms (monomial, binomial, trinomial), and the number of parts in each (1, 8, 27).

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Cubing a Trinomial Numerically

Materials:
The trinomial cube, a felt mat, tickets, prepared tickets, and pencils

Presentation:

1. Lay out the pieces of the trinomial cube in the same formation as if you had just laid out the trinomial formula
2. Review the algebraic value of the pieces: \(a^3, a^2b, a^2c, ab^2, ac^2, abc, b^3, b^2c, bc^2, \) and \(c^3\).
3. Ask the child what numbers they would like to use to work out the trinomial numerically (i.e.: 2, 3, 4).
4. Hold up the \(a^3\), ask what its value is using the numbers \((2^3)\). Have the child write a ticket and place it on the cube.
5. Continue as above with the rest. The child may use an ‘x’, a dot, or brackets to show multiplication.
6. Calculate the answer when the equation is complete. Check your work by adding ‘a’, ‘b’, and ‘c’, and cubing.

\[
(2^2, 3^2, 4^2, \quad 2\times3^2, \quad 3\times3^2, \quad 4\times3^2, \quad 2\times4^2, \quad 3\times4^2, \quad 4\times4^2)
\]

Or:

\[
\begin{array}{c}
\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times 2 \\
\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times 3 \\
\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times 4 \\
\end{array}
\]
Cubing a Trinomial Hierarchically

Note: The child must have had plenty of practice doing algebraic work before doing this exercise.

Materials:
The algebraic trinomial cube, the hierarchical trinomial cube (painted differently), a felt mat, blank tickets, prepared tickets: one set with the algebraic formula on one side and the hierarchical formula on the other, another set with ‘h’, ‘t’, and ‘u’ on one side and numbers 1,000,000 to 1 in a sequence of the intervening hierarchies on the other.

Presentation:

Passage One: The Story of the Three Kings

1. Tell the Story of the Three Kings Emphasize that they were in a parade, with each king occupying his position, when the blue king revolted, in an attempt to take the red king’s place, by sending his attendants forward.

2. Note that the red king’s bodyguards put down the insurrection, and the yellow king fled to the rear, sending his attendants to capture the blue king’s remaining men. Without knowing it, they had entered the Decimal System.

3. Lay out the algebraic terms for each piece (a^3, t^2b, etc.). Turn these over to reveal the different symbols on the backs of these.

4. State that you are trying to find out why the blue king’s attendants failed to capture the red king.

5. Ask the child what each new piece is worth using the new symbols as you turn each over (h^3, h^2t, etc.).

6. Inform the chill that the new symbols represent the decimal system: h = hundreds, t = tens, and u = units.

7. Determine the value of the pieces by matching the first set of tickets with the second that states the values (i.e.: h^3 = 1,000,000, h^2t = 100,000, etc.).

8. The child may notice that the blue king’s attendants have the same value as those of the red king.

9. She may also notice that there is a relationship as to the number of times each occurs: 1 – 3 – 6 – 7 – 6 – 3 – 1.

10. The child may wish to create a chart of these. To do so, she would need four columns: decimal category, value, number of times the value occurs and analysis.

<table>
<thead>
<tr>
<th>Decimal Category</th>
<th>Value</th>
<th># of Times Value Occurred</th>
</tr>
</thead>
<tbody>
<tr>
<td>h^3</td>
<td>1,000,000</td>
<td>1</td>
</tr>
<tr>
<td>h^2t</td>
<td>100,000</td>
<td>3</td>
</tr>
<tr>
<td>h^2t</td>
<td>10,000</td>
<td>6</td>
</tr>
<tr>
<td>h^2u</td>
<td>10,000</td>
<td></td>
</tr>
<tr>
<td>htu</td>
<td>1,000</td>
<td>7</td>
</tr>
<tr>
<td>t^3</td>
<td>1,000</td>
<td></td>
</tr>
<tr>
<td>hu^2</td>
<td>100</td>
<td>6</td>
</tr>
<tr>
<td>t^2u</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>tu^2</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>u^3</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
lay the "bodyguards" so that they are the same height or lower than their specific King.

**Initial Set Up**

- Yellow King moves to back
- Yellow King captured by yellow king attendants
- Blue King surrounded by all six bodyguards

**Ending Arrangement**

- Blue King surrounded by all six bodyguards
- Yellow King captured by yellow king attendants

---

**Story of Three Kings**
Reference:
Passage Two: The Hierarchical Trinomial

1. “There is a way that all the pieces that have the same value could have the same color.”
2. Remove the tickets. Introduce the Hierarchical Trinomial Cube. Lay the top and middle layers on the side flaps of the open box.
3. Exchange the parade. Remove the red cube, place it back in its box, and put the large blue cube in its place. Continue in this manner with the rest.
4. Note that now the kings and their attendants are orderly again, and they can all march.
5. State that this parade represents our decimal system. Point out the different categories, starting with the units.
6. Return the pieces to the box, starting with the blue cube, stating that the parade is over and the kings are going home.
The Story of the Three Kings

Once upon a time there were three kings, each the founder of a kingdom. These kings were closely allied and together formed an empire.

Each king had its own size and color.

Whenever there was a special occasion they all marched in procession with their attendants. The red king, being the largest, went first with his six attendants who dressed in the color of the king back to front. Out of the six, three were related to the blue king and the other three were related to the yellow king.

This next king, the blue king, followed with his six attendants who dressed in the color of their king back to front. Out of six, three were related to the red king and the other three to the yellow king.

Then came the yellow king with his six attendants all dressed in the color of their king back to front. Out of six, three were related to the red king and the other three to the blue king.

There were also six bodyguards dressed all over in black and related to all the kings. Each king had two bodyguards and for the sake of harmony no one could be taller than its king.

Whenever they went out, they marched in just this way -- the order and harmony was something beautiful to see. Until, one day, the blue king got tired of it -- he really didn't like being stuck in the middle. He felt he deserved better because he was the first to assume power and to start the dynasty. He was quite unlike the yellow king who was small and very modest. The yellow king was the initiator of them all but he hated making a show.

One day, the blue king started a revolt. He sent three of his attendants to capture the attendants of the red king.

There was a struggle. All the bodyguards rushed from the other kings and surrounded the blue king.

Three attendants of the yellow king moved up and captured the remaining attendants of the blue king. The three other yellow attendants moved forward as support. The yellow king himself moved to the rear to keep guard.

And so the procession continued. The red king was so mighty, he hardly knew what was going on and he went on unperturbed. He didn't even know there had been a revolt. The blue king had to move on, too -- in his same place -- this time, surrounded by the bodyguards. The yellow king walked in the rear -- quite content -- he never liked the show anyway.
Why did this happen? What caused this change of order? Well, unknown to them, they had left their own kingdoms and had entered the kingdom of the decimal system -- and their values had changed.

The red king had become the cube of hundreds, the blue king the cube of tens and the yellow king the cube of units.
IX Square Root & Cube Root

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Square Root Sensorially

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Cube Root

Passage One: Concept, Language and Notation
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Passage Four: Finding the Cube Root of Three-Digit Numbers
Passage Five: When an Incorrect Suspect Digit is Used
Passage Eight: Abstraction
Passage Nine: Formation of the Rule
Square Root Sensorially

Materials:
The bead squares from the bead cabinet, the square root board, green unit beads, the unit division cup, paper and pencils

Presentation:

Passage One: Concept, Language and Notation with the Bead Squares

1. Lay out the bead squares from one to ten. Hold up the three square, and ask what it is, how many beads are in it and how many beads are along one side.
2. State that the number along the side (3) is the square root of 9.
3. Note that it is as if the square grows up from the root, gesturing up the square from the bottom.
4. Repeat the above with the seven square and conclude that the square root of 49 is 7.
5. Have the child choose a bead square, and say what it is the square of, how many beads are in it, how many are along a side and what it’s the square root of.
6. Reveal that in Latin the word for root is radix, and write it out, noting that at first the whole word was written out to denote the square root.
7. “Later, when they did square roots, they wrote only a cursive “r” with a line following it. Eventually, they wrote this symbol: √ . Record both of these under the word radix.
8. “If you want to write the square root of this square (holding up the seven square), you would write the number of beads under the bar.”
9. Determine what you would multiply to get 49. Write this after the square root statement.
10. Have the child label each square in the same way.
Passage Two: Concept, Language and Notation with the Square Root Board

1. Present the square root board. “Let’s find out what the square root of 14 is.”
2. Place 14 beads in the unit division cup, and state that you can’t tell by looking at them.
3. “Let’s build a square on the board with the beads, starting with the square of one.” Place one bead on the board.
4. Add three beads to form the square of two, then five more to form the square of three.
5. Try to build out to the square of four, but notice that you don’t have enough beads.
6. Remove the extra beads to reform the square of three. Ask the students how they should write this. Record:
7. Have the child look to see what is left over. Have him record:

Passage Three: Finding the Square Root with the Golden Bead Material

1. “Let’s find out the square root of 576.” Bring out the golden bead material.
2. Gather the number in the material. Build the largest square you can, starting with the hundreds and exchanging as necessary.
3. Count across the bottom of the square to determine the square root (24).
4. Check by squaring 24. The child may repeat this for numbers up to 999. Beyond that, he’ll have entered another dimension.
Passage Four: Finding the Number of Digits in the Root

**Note:** You will need the square root chart of numbers from 1 to 9 for this passage.

1. Present the square root chart. “If you square a number from 1 to 9, how many digits are in the answer?”
2. Determine that if you square the numbers 1 to 9, the answer has one or two digits.
3. Note that the opposite is true as well. If a number has one or two digits, its square root will have one digit.
4. “Let’s do some work on paper. Let’s square the digits from 1 to 9 in the tens place (10, 20, 30…).
5. Determine that if you square numbers with two digits, you get answers with two or three digits. Notice that the opposite is still true.
6. Continue with other categories (100’s, 1000’s…) to discover that to determine the number of digits that will be in the square root, you mark the digits into groups of two, and count the groups, i.e.: 2’34’56 = 3 digits.
7. Propose problems to verify this. Again note that the reverse is true; if a three digit number is squared, the answer will have five or six digits.

Passage Five: Finding the Square Root with the Peg Board and Guide Squares

**Note:** You will need the peg board, pegs and guide squares for this passage.

1. Propose the problem: the square root of 625. Ask the child how many digits will be in the answer (2), have her mark the number: 6’25.
2. Lay the pegs into their corresponding cups (6 in the hundreds cup, 2 in the tens, 5 in the units).
3. Ask what the largest square you can build with the pegs is. Bring out the two digit guide square.
4. Begin by making the largest square possible with the hundreds (2²).
5. Exchange the hundreds and build off the sides with the tens.
6. Exchange the tens and fill in the rest with the unit pegs. Read across the bottom to discover that the square root of 625 is 25.
7. Have the child repeat with three or four digit numbers. She should do at least one with you supervising to see that she marks off the digits the root will have, and uses the guide squares as necessary.
Passage 5:

Square Root of 625
\( \sqrt{625} \)  How many digits? 6 25

2 left, exchange for 20 tens.

2 left, exchange for 20 units.

\( \text{answer} = 25 \)
Passage Six: Writing Answers

Part A: Remainder Only

1. Propose the problem: the square root of 576. Have the child mark off how many digits will be in the root.
2. Have the child begin to build the square with the hundreds. When it is as large as it can be, have the child write the root on paper and how many pegs were used under the hundreds. Note that you’ve used as many as possible, so you subtract it.
3. Bring down the tens on paper. Exchange the hundred peg and place the tens pegs on the board.

\[
\begin{array}{c}
576 \\
2 \\
\hline
-4 \\
17
\end{array}
\]

4. When as many ten pegs as possible have been placed on the sides of the hundreds, record how much was used up and bring down the units.

\[
\begin{array}{c}
576 \\
2 \\
\hline
-4 \\
17 \\
-16 \\
16
\end{array}
\]

5. Exchange the remaining ten and fill in the square with the unit beads. Ask the child how many beads were used up, record and subtract.
6. Ask the child what the second digit of the root is, and record it.

\[
\begin{array}{c}
576 \\
24 \\
\hline
-4 \\
17 \\
-16 \\
16 \\
-16 \\
0
\end{array}
\]
Passage Six

\[
\begin{array}{c}
\sqrt{576} \\
4 \\
17 \\
-16 \\
16 \\
-16 \\
0 \\
\end{array}
\]
Part B: Writing Answers, Amount Used and Remainder

1. Give the problem of finding the root of 54,756: have the student determine how many digits in the root and mark it.

\[ 5'47'56 \]

2. Get out the 3 digit guide square for children to use as a guide; place pegs in corresponding cups.

3. Begin building the first square with the 10,000’s. When the largest square that can be made is built (2), record the square and how many is used, subtract this:

\[
\begin{array}{c}
5'47'56 \\
14 \\
\end{array}
\]

4. Note that you have 14 to build the legs with; have the child do this, record how many is used, subtract and bring down the next category:

\[
\begin{array}{c}
5'47'56 \\
14 \\
-12 \\
27 \\
\end{array}
\]

5. Have children fill in the square, record the square (3) and how many used then subtract.

6. Continue building the legs out with the remaining pegs of that category: record how many used, subtract, bring down the next category:

\[
\begin{array}{c}
5'47'56 \\
14 \\
-12 \\
27 \\
-9 \\
18 \\
-16 \\
25 \\
\end{array}
\]

7. Continue in the same manner filling in the tens then units, subtracting the quantity used each time.

\[
\begin{array}{c}
5'47'56 \\
14 \\
-12 \\
27 \\
-9 \\
18 \\
-16 \\
25 \\
-24 \\
16 \\
-16 \\
0 \\
\end{array}
\]
Part B: Writing Answers

Square Root of 54,756
\[ \sqrt{54,756} \]
How many digits? 5 or 6

\[ \sqrt{54,756} \]
\[ \sqrt{54,756} \]
\[ \sqrt{54,756} \]
\[ \sqrt{54,756} \]

- 4
- 4
- 4
- 4

- 14
- 14
- 14
- 14

- 12
- 12
- 12
- 12

- 27
- 27
- 27
- 27

- 25
- 25
- 25
- 25

- 2
- 2
- 2
- 2

R
exchange

R
exchange

R
exchange

R
exchange

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Part B: Writing Answers:

Cont’d from above.
Part C: Writing Answers with an Analysis of Amount Used

1. Propose the problem: the square root for 27,394,759; determine digits (4); get the appropriate guide square.
2. Begin the problem in the same way; find the first square, record it to the right of the problem, write how many were used and subtract.
3. Ask the child how you could write the root of 25 ($5^2$) and have the child do so to the left of the 25:
   
   $27'39'47'59 \quad 5^2$
   
   $-25$
   
   23

4. Continue working the problem in the same way working out the components each time a quantity is used up:
Passage Seven: Special Cases

Note: Materials are the same as for squaring with pegs plus marker strips for the peg board.

Part A: Zero in the Middle of the Root

1. Find the square root of 91,204: begin building the first square in the same way as above.
2. Note on the guide square the parts missing (only one green peg which isn't enough to build the legs out).
3. Introduce the marker strips which hold the category; place them criss-crossed on the board and record zero in the answer.
4. Continue the problem in the manner described above.

\[
\begin{array}{c}
9'12'04 \\
-9 \\
-12 \\
-004 \\
\end{array}
\]

\[
\begin{array}{c}
302 \\
12 \\
04 \\
0 \\
\end{array}
\]

Part B: Zero at the End of the Root

1. Use a problem such as finding the square root of 102,400 to show this.
2. Begin and work in the same way as above. Refer to guide square, noting that a category is missing at the end.
3. Place the marker strips to hold the space; record zero as the last digit of the root.

\[
\begin{array}{c}
10'24'00 \\
-9 \\
-12 \\
04 \\
\end{array}
\]

\[
\begin{array}{c}
320 \\
12 \\
000 \\
\end{array}
\]
Passage 7: Part A

Square Root of 91,204
\[ \sqrt{91,204} \text{ How many digits?} \]

\[ \sqrt{91,204} = 302 \]

IX_A_7_A

\[ \sqrt{102400} = 320 \]
Square Root Leading to Abstraction

Materials: Same as squaring sensorially with pegs.

Presentation:

Passage One: Building the Square by Periods

1. Propose the problem: find the square root of 54,756. Get the guide square and set up as usual.
2. Build the first square, record the root and how many were used up, subtract.
3. Before bringing the next digit down, have the child look at the guide square, ask her what categories will be brought down (1,000’s on one side, fill in with 100’s).
4. Bring down both on paper and **build out in an inverted “L” around the square**, filling in both the leg and square at the same time.

5. When completed as far as possible, record the square and determine how many were used up (120 + 9 = 129), subtract and bring down the next categories (10’s and units).
6. Continue building in the same fashion, completing the square each time.
7. When complete, count up and record how many pegs were used and the root of the square.
8. Check beads to see if they correspond with the written answer.
Passage 1: Building Square by Periods
Passage Two: Building the Square with Calculations

1. Using the same problem, work out to the point of recording the first square and the amount used. Subtract this and bring the next categories down.

2. Have the child look at the guide square. State that you'll build out on 2 sides, ask what categories you'll use (1,000's and 100's).

3. Note that every time you build out 1 space, you'll have to place 4 pegs.

4. State that you have 14 and place a comma after this in the problem. Calculate how far you'll go out by dividing the 14 by 4 and rounding to 3.

5. Build out with both categories in the manner utilized in the preceding problem, record the root, how much was used and subtract.

6. Place a comma between the 5 and 6; note that you'll need 23 on each side to build out the legs or 46 all together.

7. Divide the 185 by 46 to see how far you'll have to build out (4), record this to the left of the 1856.

8. Lay out the rest of the pegs completing the problem in the same manner as before.
Passage Three: Calculation of Square Root on Paper

1. Repeat the problem above on paper, using the peg board to check - see Passage One above for reference.
2. Have the child find the square root of the first number (she should know it in her head by now).
3. Record in the same way placing the root to the right of the problem, it’s value under the first digit of the number and subtracting it.

\[ \sqrt{547562} \]

- 4

\[ \sqrt{547562} \]

14 \div 4 = 3r2

147

\[ 147 \]

- 129

\[ 129 \]

43 \times 3 = 129

(40 \text{hundreds} \times 3) + (3 \text{hundreds} \times 3) = (43 \times 3)

4. Ask what categories are next (1,000’s and 100’s) bring these down.
5. Ask how many 1000s you’ll place for each row (4); divide this into the first 2 digits of the remainder for a trial root:

\[ \sqrt{547562} \]

- 4

147

\[ 147 \]

- 129

\[ 129 \]

14 \div 4 = 3r2

43 \times 3 = 129

6. Note if you build out 3, you’ll have 4 1000’s times 3 for the legs, and 3 across the hundreds square (also times 3). Put these numbers together and multiply by the root for the total of what’s been used up:

\[ \sqrt{547562} \]

- 4

147

\[ 147 \]

- 129

\[ 129 \]

14 \div 4 = 3r2

43 \times 3 = 129

(40 \text{hundreds} \times 3) + (3 \text{hundreds} \times 3) = (43 \times 3)

7. Subtract this amount after recording the amount; bring down the next categories (1856).
8. Divide the first 3 digits of this number by how many times you’ll have to build out (double the root found so far: 23 \times 2 = 46).
9. Add the 46 tens from the legs in the square to the 4 units of the final square and multiply by 4; subtract this number (1856) and record the root:
The Rule of Extraction Of Square Roots

1. Point off as many periods of two digits each as possible starting from the units.
2. Find the greatest number whose square is equal to or less than the left hand period and write this number as the first digit of the root.
3. Square the first digit of the root, subtract its square from the first period, and bring down the second period to the remainder.
4. Double the part of the root already found for a trial divisor, divide it into the remainder (omitting from the latter the right hand digit) and write the quotient as the next digit of the root. 141/2 4=3
5. Bring down the root digit (3) just found to the trial divisor (4) to make the complete divisor, multiply the complete divisor by this root digit, subtract the result from the dividend (129) and bring down to the remainder the next period for a new dividend.
6. Double the part of the root already found for a new trial divisor (46) and proceed as before until the desired number of digits of the root have been found. Square the root and any remainder found to check.

First Course in Algebra, Hawkes-Luby-Touton, p.233
Cube Root

Prerequisite: The child should have completed all the squaring and cubing work up through the hierarchic trinomial. She should be facile at writing square roots, but she may not yet have learned the rule.

Passage One: Concept, Language and Notation

Materials: Cubes from the bead cabinet, paper and pencil, and 2 cm cubes.

Presentation:
1. Hold up a cube of 4; ask what it is, how many beads are in it.
2. State that the cube root of 64 is 4 pointing out the root along all three dimensions while doing so.
3. Either do another or have a child in the group do one; have children take turns doing it for all the bead cubes.
4. State that this is called the cube root and to notate it you draw the frame as for square root and add a small 3 as follows:
5. Write out a cube to show the children; allow children to write out the cube root of the other cubes.
6. State that these aren't the only numbers with cubes. Write out:

\[ 3^3 \sqrt{35} = \]

7. Note that there is no cube for this, but you'd still like to find the cube root; ask the children if they can think of a way to do this.
8. Introduce the 2cm cubes; take out 35 cubes and have children build a cube of one then continue to build in sequence building out by dimensions and filling in until the largest cube that can be built is made.
9. Show children how to write this noting you can read the cube root along the edge and then count those left for the remainder:
10. Have each child in the group build one or two of these; children will become aware that any number contains a root and the remainder can be larger than the cube root.
Passage Two: Using the Chart to Find Single Unit Cube Roots

Materials: Cube chart consisting of 4 columns, one of numbers 1-9, followed by the square of these numbers, then their cubes, then the number multiplied 3 times and squared.

Presentation:
1. Present the chart to the children; review that for a 2 digit number, the square root was one digit.
2. Note that in a cube, you get a one digit root if the number has up to 3 digits. You have to allow 3 digits for every digit of the root.
3. State that you can use this chart to find the cube root for any number with 3 digits.
4. Pick a number to see how this works i.e. 316; look for the number on the chart.
5. Note that it’s not there, but it is between 216 and 343 (cubes of 6 and 7) so the largest cube built would be 6.
6. Show the children how to calculate the remainder by subtracting this number (216) from the number you’re trying to find the root for. (6 r100).
7. Children can work with the chart to figure out the cube root of other numbers.

<table>
<thead>
<tr>
<th>N</th>
<th>N^2</th>
<th>N^3</th>
<th>3N^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
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<td>2</td>
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<td>8</td>
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</tr>
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<td>108</td>
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<td>7</td>
<td>49</td>
<td>343</td>
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<tr>
<td>8</td>
<td>64</td>
<td>512</td>
<td>192</td>
</tr>
<tr>
<td>9</td>
<td>81</td>
<td>729</td>
<td>243</td>
</tr>
</tbody>
</table>
Passage Three: Finding the Number of Digits and the Cube Root of Two-Digit Numbers

Materials: Cube root, and wooden cubing material.

Presentation:
1. May begin by referring to a question a child may have had concerning the last column on the chart.
2. State that you want to see what happens when we have more than 3 digits in a number and we want to find its cube root.
3. Give the problem of finding the cube root for 79,507 and write this out.
4. Ask the child how many digits will be in it; have them mark it:
\[ \sqrt[3]{79'507} \]

5. Have the child look at the first period and determine its cube root; ask how much was used up, record as follows:
\[
\begin{array}{c}
(4^3) \\
\sqrt[3]{79'507} \\
- 64 \\
15
\end{array}
\]

6. State that we will use the cubing material to keep track; bring out the 4 cube for this (represents \(4^3\)).
7. State that we have another digit to find and will use the 4 cube to find it.
8. State that you want to see how large the next part of the root will be; ask what will be needed to build out each time.
9. Note that you will have to build out with 4 squares and will have to do so in three directions, note where this is on the chart and calculate: \(n=4, \text{ so } 3n^2 = 3(4^2) = 48\).
10. State that this is the next category; divide what is left by this number to find your probable unit root.
11. Record and calculate:
\[
\begin{array}{c}
(4^3) \\
\sqrt[3]{79'507} \\
- 64 \\
155 \\
\div 48 = 3
\end{array}
\]
12. Ask **what would happen if we built out 3**, how much was used up; subtract what is used up; ask what else is needed to build, bring down the next category:

13. Build out from the cube 3 places in 3 directions:

\[
\begin{array}{c}
3 \sqrt{79'507} \\
(4^3) \\
155 \div 48 = 3 \quad 48 \times 3 = 144 \\
\hline
155 \\
-64 \quad 144 \\
\hline
110 \\
\end{array}
\]

14. Determine that you need to fill in with 3 groups of 3 squares 4 times; record this to the left of the problem and calculate; bring down the next category to see if you’ll have enough:

\[
\begin{array}{c}
3 \sqrt{79'507} \\
(4^3) \\
155 \div 48 = 3 \quad 48 \times 3 = 144 \\
\hline
155 \\
-64 \quad 144 \\
\hline
110 \\
\end{array}
\]

\[
3 \times (3^2 \times 4) \\
\hline
108 \\
\hline
27 \\
\end{array}
\]

15. Determine numerically if you have enough to build the cube and record:

\[
\begin{array}{c}
3 \sqrt{79'507} \\
(4^3) \\
155 \div 48 = 3 \quad 48 \times 3 = 144 \\
\hline
155 \\
-64 \quad 144 \\
\hline
110 \\
\end{array}
\]

\[
3 \times (3^2 \times 4) \\
\hline
108 \\
\hline
27 \\
\end{array}
\]

16. Finish the cube and cube 43 to check.

\[
3 \times (3^2 \times 4) \\
\hline
27 \\
\hline
27 \\
\end{array}
\]

\[
3^3 \\
\hline
0 \\
\end{array}
\]
Passage Four: Finding the Cube Root of Three-Digit Numbers

Materials: Hierarchical Trinomial (h = blue, t = yellow, u = red)

Presentation:

1. Give the problem of finding the cube root of 143055667; have the children write this out and mark how many periods there are.
2. State that you will use the hierarchical trinomial to work it out; have the children lay out the pieces of it in their categories.
3. Begin in the same way finding the cube of the first period; determine that the cube root for 143, subtract and bring down the next category.
4. Record the 5 in the hundreds spot and under this write 53 = 125; record the 125 under 143, subtract and bring down the next category.
5. Take out the blue cube to represent hundreds and record h3 (hundreds cubed) to the left of the problem:

\[
\sqrt[3]{143'055'667} = 5 \\
\begin{array}{c}
5^3 = 125 \\
\hline
180
\end{array}
\]

6. Determine that to build the next three dimensions, you’ll have to take 52 3 times (75); divide what we have (180) by this to get our next suspect digit of the cube root. (2)
7. Calculate the algebraic and numerical values and record; build the next category of prisms onto the cube and subtract:

\[
\sqrt[3]{143'055'667} = 5 \\
\begin{array}{c}
5^3 = 125 \\
\hline
180
\end{array}
\]

\[
\begin{array}{c}
3h^2t = 3(5^2 \times 2) = 150
\end{array}
\]
8. Bring down the next category of prisms and the next digit in the number to get 305 from which to build the section.

9. Calculate that the next prisms equal $3ht^2$; plug in the numbers and see if there is enough: $3(5 \times 22) = 60$. Record and subtract: (Build prisms onto cube)

\[
\begin{array}{c}
\sqrt[3]{143'055'667} \\
\hline
h^3 & 5^3 = 125 \\
12.5 & \\
18.0 & 180 \div 75 = 2 \\
3h^2t & 3(5^2 \times 2) = 150 \\
15.0 & \\
3.05 & \\
3ht^2 & 5(5 \times 2^2) = 60 \\
-60 & \\
2.45 & \\
3h^2u & 3(5^2 \times 3) = 225 \\
-225 & \\
20.5 & \\
\end{array}
\]

10. Bring down the next category of prisms; note that there is no value for the units and to find this we'll have to go out 5 square 3 times.

11. Calculate this amount $5^2 \times 3 = 75$; divide the remaining amount by this to get the suspect unit number: $245 \div 75 = 3$.

12. Work out for the exact amount; $3(5^2 \times 3) = 225$, and subtract: (Build prisms onto cube)
13. Bring down the next category and the tens cube; confirm the suspect number for the tens as a digit of the root by cubing it.

14. Determine that it works, record all values calculated and subtract the cube value.

\[ \sqrt[3]{143'055'667} = 5 \]

\[
\begin{array}{c}
\h^3 \\
-125 \\
+180 \\
5^3 = 125 \\
\end{array}
\]

\[
\begin{array}{c}
3\h^2t \\
-150 \\
+305 \\
3(5^2 \times 2) = 150 \\
\end{array}
\]

\[
\begin{array}{c}
3\h t^2 \\
-60 \\
+245 \\
5(5 \times 2^2) = 60 \\
\end{array}
\]

\[
\begin{array}{c}
3\h^2u \\
-225 \\
+205 \\
3(5^2 \times 3) = 225 \\
\end{array}
\]

\[
\begin{array}{c}
t^2 \\
-8 \\
+197 \\
2^3 = 8 \\
\end{array}
\]

15. Calculate the algebraic and numerical values of the next group of prisms, record and build onto the cube:

\[ \sqrt[3]{143'055'667} = 5 \]

\[
\begin{array}{c}
\h^3 \\
-125 \\
+180 \\
5^3 = 125 \\
\end{array}
\]

\[
\begin{array}{c}
3\h^2t \\
-150 \\
+305 \\
3(5^2 \times 2) = 150 \\
\end{array}
\]

\[
\begin{array}{c}
3\h t^2 \\
-60 \\
+245 \\
5(5 \times 2^2) = 60 \\
\end{array}
\]

\[
\begin{array}{c}
3\h^2u \\
-225 \\
+205 \\
3(5^2 \times 3) = 225 \\
\end{array}
\]

\[
\begin{array}{c}
t^2 \\
-8 \\
+197 \\
2^3 = 8 \\
\end{array}
\]

\[
\begin{array}{c}
6\h t u \\
-180 \\
+176 \\
6(5 \times 2 \times 3) = 180 \\
\end{array}
\]
16. Continue in the same manner determining the algebraic values of the prisms (and the last cube) and plugging in the numerical values: (Building onto the cube after each)

\[ \sqrt[3]{143'055'667} \]

<table>
<thead>
<tr>
<th>Term</th>
<th>Calculation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h^3 )</td>
<td>(-125)</td>
<td>(5^3 = 125)</td>
</tr>
<tr>
<td></td>
<td>(+180)</td>
<td>(180 ÷ 75 = 2)</td>
</tr>
<tr>
<td>( 3h^2t )</td>
<td>(-150)</td>
<td>(3(5^2 \times 2) = 150)</td>
</tr>
<tr>
<td></td>
<td>(+305)</td>
<td>()</td>
</tr>
<tr>
<td>( 3ht^2 )</td>
<td>(-60)</td>
<td>(5(5 \times 2^2) = 60)</td>
</tr>
<tr>
<td></td>
<td>(+245)</td>
<td>()</td>
</tr>
<tr>
<td>( 3h^2u )</td>
<td>(-225)</td>
<td>(3(5^2 \times 3) = 225)</td>
</tr>
<tr>
<td></td>
<td>(+205)</td>
<td>()</td>
</tr>
<tr>
<td>( t^2 )</td>
<td>(-8)</td>
<td>(2^3 = 8)</td>
</tr>
<tr>
<td></td>
<td>(+197)</td>
<td>()</td>
</tr>
<tr>
<td>( 6htu )</td>
<td>(-180)</td>
<td>(6(5 \times 2 \times 3) = 180)</td>
</tr>
<tr>
<td></td>
<td>(+176)</td>
<td>()</td>
</tr>
<tr>
<td>( 3hu^2 )</td>
<td>(-135)</td>
<td>(3(5 \times 3^2) + 135)</td>
</tr>
<tr>
<td></td>
<td>(+41)</td>
<td>()</td>
</tr>
<tr>
<td>( 3t^2u )</td>
<td>(-36)</td>
<td>(3(2^2 \times 3) = 36)</td>
</tr>
<tr>
<td></td>
<td>(+56)</td>
<td>()</td>
</tr>
<tr>
<td>( 3tu^2 )</td>
<td>(-54)</td>
<td>(3(2 \times 3^2) = 54)</td>
</tr>
<tr>
<td></td>
<td>(+27)</td>
<td>()</td>
</tr>
<tr>
<td>( u^3 )</td>
<td>(-27)</td>
<td>(3^3)</td>
</tr>
<tr>
<td></td>
<td>(+0)</td>
<td>()</td>
</tr>
</tbody>
</table>

The answer can be checked by cubing 523.
Passage Five: When an Incorrect Suspect Digit is Used

1. Work out in the same way as in example one except when the suspect digit doesn’t work (there is not enough to make the next category with), erase this part (shown in the brackets) and continue substituting a digit one number smaller:

\[
\sqrt{647214625} = 865
\]

<table>
<thead>
<tr>
<th>Digit</th>
<th>Calculation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>h³</td>
<td>83 = 512</td>
<td></td>
</tr>
<tr>
<td>3h²t</td>
<td>1352 ÷ 192 = [7] [6]</td>
<td>3(8²x7) = 1344</td>
</tr>
<tr>
<td>3h²t</td>
<td>1352</td>
<td>3(8²x6) = 1152</td>
</tr>
<tr>
<td>3ht²</td>
<td>1176</td>
<td>3(8x6³) = 864</td>
</tr>
<tr>
<td>3h²u</td>
<td>960</td>
<td>3(8² x 5) = 960</td>
</tr>
<tr>
<td>t³</td>
<td>216</td>
<td>6³ = 216</td>
</tr>
<tr>
<td>6htu</td>
<td>1440</td>
<td>6(8 x 6 x 5) = 1440</td>
</tr>
<tr>
<td>3hu²</td>
<td>600</td>
<td>3(8 x 5³) = 600</td>
</tr>
<tr>
<td>3t²u</td>
<td>540</td>
<td>(6² x 5) = 540</td>
</tr>
<tr>
<td>3tu²</td>
<td>450</td>
<td>3(6 x 5³) = 450</td>
</tr>
<tr>
<td>u³</td>
<td>125</td>
<td>5³ = 125</td>
</tr>
</tbody>
</table>
Passage Six: Three-Digit Root with a Zero in the Middle

1. Begin to work out in the same way; when zero comes out to be the trial digit for the tens, work out in the same way as if it were a number putting away the cubes not used because their values came out to zero.

\[
\sqrt[3]{64'964'808} = 402
\]

\[
\begin{array}{c}
\text{h}^3 \\
- \frac{64}{0} \\
9 \\
\end{array}
\]

\[
\begin{array}{c}
4^3 = 64 \\
9 \div 48 = (0)
\end{array}
\]

\[
\begin{array}{c}
3\text{h}^2\text{t} \\
- 0 \\
96 \\
\end{array}
\]

\[
\begin{array}{c}
3\text{h}^2\text{u} \\
- 96 \\
04 \\
\end{array}
\]

\[
\begin{array}{c}
3(4^2 \times 2) = 96
\end{array}
\]

\[
\begin{array}{c}
\text{t}^3 \\
- 0 \\
4 \\
\end{array}
\]

\[
\begin{array}{c}
6\text{htu} \\
- 0 \\
48 \\
\end{array}
\]

\[
\begin{array}{c}
3\text{h}\text{u}^2 \\
- 48 \\
0 \\
\end{array}
\]

\[
\begin{array}{c}
3(4 \times 2^3) = 48
\end{array}
\]

\[
\begin{array}{c}
3\text{t}^2\text{u} \\
- 0 \\
00 \\
\end{array}
\]

\[
\begin{array}{c}
3\text{tu}^2 \\
- 0 \\
08 \\
\end{array}
\]

\[
\begin{array}{c}
u^3 \\
- 8 \\
0 \\
\end{array}
\]

\[
2^3 = 8
\]

The formation of the built cube should be a binomial when completed.
Passage Seven: Three-Digit Root with a Zero at the End

1. Work out in the same manner finding zero as the trial root for the units; work out until the cube then children should realize the rest will be zero because the unit is zero. Put away the pieces not used; the built cube should be a binomial when completed.

\[
\sqrt[3]{79'507'000} = 430
\]

\[
\begin{align*}
\hbar^3 & - 64 \\
& 155 \\
3\hbar^2t & - 144 \\
& 110 \\
3\hbar t^2 & - 108 \\
& 2 \\
t^3 & - 27
\end{align*}
\]

Passage Eight: Abstraction

1. To do it without the material, have children record the problem, mark off how many digits and calculate the first cube in the same way:

\[
\sqrt[3]{80'621'528} = 4
\]

\[
\begin{align*}
\hbar^3 & - 64 \\
& 16621 \\
(h + t)^3 & 79507 \\
& 1114
\end{align*}
\]
6. Bring down the next period; mark off the first category of this with a comma (11145,28).
7. Divide the quantity to the left of the comma by 3 times the first 2 digits squared \(3(43^2)\) to get the next trial digit: \(11145 \div 5547 = (2)\).
8. Cube all three digits \((h + t + u)^3\), subtract this from the total number to confirm all the digits.
9. If the number is too large, reduce it by one unit and try again.

\[
\begin{array}{c|c}
\text{h}^3 & 80'621'528 \\
\hline
\text{h}^3 - 64 & 4 \\
16621 & 4^3 = 64 \\
\text{(h + t)}^3 & 166 \div 48 = (3) \\
79507 & (43^3) = 79507 \text{ [take 3 times]} \\
11145,28 & 11145 \div 3(43^2) = \\
(\text{h + t + u})^3 & 80621528 \\
& 11145 \div 5547 = (2) \\
& 0
\end{array}
\]

\[80621 - 79507 = 1114\]

**Passage Nine: Formation of the Rule**

1. Divide the number into the period of 3 digits beginning with units and proceeding to the left.
2. Extract the cube root of the first period, this will be the first digit of the root.
3. Cube this number and subtract from the first period.
4. To this remainder, bring down the first digit of the next period.
5. Divide this number by 3 times the square of the root so far discovered. This quotient is the next trial digit of the root.
6. Cube the entire root so far discovered (including the new trial digit) and subtract this from the period involved. If the number is too large, reduce the trial figure by one and try again.
7. Repeat from step 4 until all digits are found.